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AN ELEMENTARY TREATISE ON ALTERNATING CURRENTS

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AN

# ELEMENTARY TREATISE

ON

# ALTERNATING CURRENTS



BY

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# PREFACE

Having been for some years engaged in lecturing to advanced students on the subject of alternating currents, the author has endeavoured to present, in as simple a manner as possible, a subject which, though intensely interesting in its purely physical aspect, is often expounded by mathematicians who, while revelling in the mathematical gymnastics afforded by its problems, so bewilder their less mathematical readers by the adoption of unnecessarily ponderous methods that the majority give up in disgust a subject which would otherwise exercise great fascination for them. That such should ever happen is lamentable in the extreme, since it retards the progress of the practical application of alternating currents.

The author has employed the method of vector algebra wherever possible to solve the various problems, as he has found that it is quite easy for a student knowing only the elements of algebra and trigonometry to obtain a good working knowledge of the method in a very short time, and become enabled to attack problems which are otherwise beyond his comprehension.

The application of vector algebra to alternating-current problems should appeal to everybody. It is a tool peculiarly adapted to the subject, and combines simplicity with all the advantages of a powerful method.

No one will dispute the use of a treatise on the subject in which the chief aim of the writer is to eliminate mathematical difficulties and give prominence to the physics of the subject. Since the author has engaged in practice as a consulting engineer he has found that the want of a simple though comprehensive treatise is often a deterrent to practical engineers acquiring a requisite knowledge of the subject.

Since the principles of continuous currents became intelligible to practical engineers, their commercial application has gone rapidly forward; it only requires the principles of alternating currents to be similarly placed before them for a like increase in their application to practice to result.

In the body of the book every effort has been made to eliminate mathematical difficulties, only a simple differentiation or integration being occasionally used. Those readers who are conversant with higher mathematics will find in the Appendix proofs of results which are assumed in the text, and also a few more difficult problems, some of which are of greater theoretical than practical interest.

The author hopes that his efforts will be appreciated by practical engineers, University honours students, and the more advanced students in Technical Schools studying for the Honours Grade in Electric Lighting for examinations of the City and Guilds of London Institute. It is to be emphasized that a short course of Vector Algebra should form a portion of the curriculum of every University College and Technical Institute.

As the book has been written in spare moments since leaving the teaching profession, the author has not been able to acknowledge all the various sources from which information has been obtained. The work is really a systematic arrangement of lecture notes compiled during the last ten years, and which contain information gathered from text-books, periodicals, and the journals of various learned societies, as well as from the writings of the author himself. It is, however, impossible to over-estimate the indebtedness to the writings of Professor S. P. Thompson and Mr. C. P. Steinmetz. It would, in fact, be impossible to write a treatise on alternating currents without drawing much from their valuable contributions to the subject.

The author has to thank Messrs. Ferranti Ltd., Messrs. Witting Brothers, Ltd., and The British Thomson-Houston Company, Ltd., for their kindness in giving him information respecting machines manufactured by them; his friends Dr. C. H. Lees of Owens College, Manchester, Mr. T. Mather of the

Central Technical College, London, and Miss Patterson, of the Manchester High School, for many suggestions and for reading the proofs, and also one of his assistants, Mr. W. Hyde, for preparing the figures and drawings.

It is to be hoped that no important error has escaped notice. Should, however, any reader detect any errors, the author will be glad if he will be good enough to point them out.

W. G. RHODES.

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1902.

## CONTENTS

# ERRATA.

Page 16, line 15, *for* "$M\frac{di_2}{dt}$" *read* "$M\frac{di_1}{dt}$."

" 38, line 21, *for* "$E$ and $I$" *read* "$V$ and $A$."

" 38, line 27, *for* "$E$ and $I$" *read* "$V$ and $A$."

" 38, last line but one, *for* "$EI \cos\theta$" *read* "$VA \cos\theta$."

" 38, last line, *for* "$EI$" *read* "$VA$."

" 39, line 17, *for* "$EI \cos\theta$" *read* "$VA \cos\theta$."

" 50, line 25, *for* "$Oy'_1$" *read* "$Oy'$."

" 56, last line but one, *for* "$k = -1$" *read* "$k^2 = -1$."

" 58, line 15, *for* "§§ 38 and 40" *read* "§§ 38 and 41."

# AN ELEMENTARY TREATISE

ON

# ALTERNATING CURRENTS

## CHAPTER I.

Introductory—Law of Force—Unit Pole—Strength of Magnetic Field—Unit Current—Magnetic Field due to Solenoid—Distinction between Induction and Magnetizing Force—Magnetic Field due to an Electric Current—Self and Mutual Induction—Potential—Specific Inductive Capacity—Equipotential Surfaces—Capacity—Condensers—Energy of a charged Condenser.

### INTRODUCTORY.

**1. Magnetic Force.**—If a magnet is placed anywhere in space and a pivoted magnet or compass needle is placed near it, the latter always takes up a definite position relative to the former. The magnet exerts a force on each end of the compass needle, which has a definite direction at every point at which the needle is placed, and the magnitude of the force is the same as if two equal quantities, $+m$ and $-m$, of magnetism (whatever magnetism may be), but of opposite sign, were situated at definite points within the magnet. These two points are called the poles of the magnet, and $m$ is called the **Pole Strength.**

If we could isolate one of the poles of the compass needle we should find that at every point in space it would be urged in a definite direction, due to the action of the magnet, and if it were always moved in the direction in which the magnet urged it, a definite curve would be traced, the tangent at every point of which would give the direction of the force at that point.

B

Such a curve is called a **Line of Force,** and the whole space under the influence of the magnet is called its **Field of Force.**

**2. Law of Force.**—If two magnet poles of strengths $m$ and $m'$ are situated at a distance $r$ apart, they exert a mutual force $F$, which is always given by the equation—

$$F = \frac{kmm'}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

where $k$ is a constant depending on the medium and the units chosen. The force is attractive if $m$ and $m'$ have opposite signs, and repulsive if they have like signs. The direction of the force is in the straight line joining the poles.

We choose our **unit pole** to be such that if two unit poles are placed at a distance of one centimetre apart in air, the mutual force exerted is equal to one dyne. The constant $k$ then becomes unity for air, and equation (1) takes the simpler form—

$$F = \frac{mm'}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If a magnetic field is due not to a single pole, but to any distribution of poles, the force exerted upon a magnet pole placed in it is still perfectly definite in magnitude and direction at every point. The magnitude of the force exerted by a magnetic field upon unit positive pole placed at any point is called the **Strength of the Field** at that point, and the direction of the field is taken to be that in which the unit positive pole is urged. The direction of the field, then, is given at every point in space by the line of force through that point. By a suitable convention the strength of the field at every point can also be represented by means of lines of force.

**3. Convention for measuring Strength of Field.**—The convention which is adopted in order to effect a representation of the strength of a magnetic field is the assumption of a uniform radiation of $4\pi m$ lines of force from a pole of strength $m$.

Equation (2) shows that the force at a distance $r$ from a pole of strength $m$ is $\frac{m}{r^2}$.

If we suppose that such a pole is situated at the centre of

a sphere of radius $r$ centimetres, the number of lines of force crossing each square centimetre of surface on the sphere is—

$$\frac{4\pi m}{4\pi r^2}, \text{ or } \frac{m}{r^2}$$

that is, the force at any point on the surface of the sphere is numerically equal to the number of lines cutting the sphere per square centimetre of surface.

It is to be noticed that these lines of force cut the surface of the sphere normally. By this convention, therefore, we can say that the strength of a magnetic field at any point is measured by the number of lines of force passing normally through a square centimetre with the given point at its centre.

The number of lines of force per square centimetre passing normally through a surface is called the **Induction** through the surface.

The total number of lines of force passing through a surface is called the **Total Flux** through the surface.

**Uniform Field.**—If the lines of force in a given region are everywhere parallel to the same direction, and if the induction at every point is the same, the field of force is said to be **uniform.**

**4. Magnetic Field due to an Electric Current.**—When an electric current flows along a conducting wire a magnetic field is produced, the lines of force of which are closed curves, some within the substance of the wire and the others threading through the circuit.

**Unit Current.**—The unit of current is defined to be that current which, flowing in a conducting wire one centimetre long bent into the arc of a circle of one centimetre radius, acts on unit magnetic pole placed at the centre with a force equal to one dyne.

The practical unit of current—the **Ampere**—is one-tenth of the C.G.S. unit of current thus defined.

**Strength of Field inside a Solenoid.** — The magnetic field inside a helix of many turns carrying a current of $i$ C.G.S. units is practically uniform at points at a moderate distance from the ends, and its strength is equal to $4\pi in$, where $n$ is the number of turns per centimetre measured along the axis of the helix. If the current is $i$ amperes, the strength of the magnetic field is given by—

$$F = \frac{4\pi in}{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

A helix of many turns is sometimes called a **Solenoid.**

### Distinction between Induction and Magnetizing Force.

**5.** When iron is placed in a magnetic field it becomes magnetized. When soft iron is freshly annealed its magnetism disappears if on being removed from the magnetizing field of force it is subjected to mechanical vibration.

Iron possesses the property of causing the lines of force to pass through it in preference to the surrounding air—or whatever the surrounding medium may be.

That which magnetizes the iron—or the **Magnetizing Force**—is the strength of the magnetic field in which it is placed; but the number of lines passing normally through each square centimetre of the iron, or the **Induction** in the iron, is many times greater than the strength of the field in which the iron is placed.

If we represent the field strength by $H$ and induction by $B$, then $H$ and $B$ are identical when the medium in which $B$ is measured is non-magnetic. When, however, $B$ is measured in a magnetic medium such as iron, we have the relation $B = \mu H$.

$\mu$ is a variable quantity depending upon the value of $B$, and is called the **Permeability** of the iron, and for moderate values of $B$ is very large; *e.g.* in annealed wrought iron when $H = 1{\cdot}6$ units, $B = 5000$ (about), so that $\mu = 3000$. The following table gives the values of $B$ and $\mu$ for different values of $H$ for Swedish iron and grey cast iron.

TABLE I.

| Swedish Iron. | | | Grey Cast Iron. | | |
|---|---|---|---|---|---|
| $H$. | $B$. | $\mu$. | $H$. | $B$. | $\mu$. |
| 0·79 | 1,500 | 1,900 | 5 | 4,000 | 800 |
| 0·90 | 2,000 | 2,200 | 10 | 5,000 | 500 |
| 1·15 | 3,000 | 2,600 | 21 | 6,000 | 279 |
| 1·38 | 4,000 | 2,900 | 42 | 7,000 | 133 |
| 1·67 | 5,000 | 3,000 | 80 | 8,000 | 100 |
| 2·03 | 6,000 | 2,950 | 127 | 9,000 | 71 |
| 2·48 | 7,000 | 2,820 | 188 | 10,000 | 53 |
| 3·00 | 8,000 | 2,670 | 292 | 11,000 | 37 |
| 3·67 | 9,000 | 2,450 | | | |
| 4·55 | 10,000 | 2,200 | | | |
| 8·00 | 12,000 | 1,500 | | | |
| 17·50 | 14,000 | 800 | | | |
| 35·56 | 16,000 | 450 | | | |

## Magnetic Field due to a Current.

**6.** When an electric current flows in a closed circuit consisting of a single loop, a magnetic field is formed, and all the lines of magnetic force either thread through the circuit or are within the wire itself. Moreover, the number of lines interlinking with the circuit is proportional to the current flowing in the circuit, provided the permeability of the medium in which the circuit is placed is constant. If the total number of lines passing through the circuit is $N$ when a current of $i$ C.G.S. units is flowing in it, we then have—

$$N = Li. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $L$ is a constant depending only on the geometrical form of the circuit.

$L$ is called the **Self-induction,** or simply the **Self-inductance** of the circuit, and is the total flux of magnetic lines through the circuit when unit current flows round it.

If, however, the closed circuit consists of a series of loops, so that each line of magnetic force may encircle the circuit more than once, the self-induction of the circuit is the sum of the products of each line of force multiplied by the number of times it would cut the circuit while being completely withdrawn when unit current is flowing through the circuit.

**7.** If there are two neighbouring closed circuits, one of which carries unit current while no current flows round the other, the number of lines of force interlinking the second circuit due to the unit current in the first is called the **Mutual Induction,** or **Mutual Inductance,** of the two circuits, and is denoted by the letter M. It is, however, to be noticed again that if a line of force due to the current in one circuit encircles the other circuit $n$ times, it must be reckoned $n$ times over, since its effect is the same as that of $n$ lines of force encircling the circuit once. It may be proved that[1]—

$$M = \int\int\frac{\cos \varepsilon \, ds \, . \, ds'}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $ds$ and $ds'$ are elementary lengths of the two circuits, distant $r$ centimetres apart, and $\varepsilon$ is the angle between the tangents to the

[1] Maxwell's "Electricity and Magnetism," vol. ii. pp. 46 and 151, third edition, 1892.

respective circuits at $ds$ and $ds'$, the integration extending round both circuits.

The above expression for $M$ shows that it depends simply upon the geometrical configuration of the two circuits, and that the relation is reciprocal.

It follows that if the two circuits carry currents $i$ and $i'$ respectively, the number of lines of force interlinking both circuits is $M(i + i')$.

**8.** If two quantities, $q_1, q_2$, of electricity are situated at a distance $r$ apart, the mutual force exerted between them is given by—

$$F = \frac{kq_1q_2}{Kr^2} \quad \ldots \ldots \ldots \quad (6)$$

where $k$ is a constant depending upon the units chosen, and $K$ is a constant depending upon the intervening medium, and is called the **Specific Inductive Capacity** of the medium: it is taken to be unity when the medium is air. The force is attractive if $q_1$ and $q_2$ have opposite signs, and repulsive if they have like signs.

**Electrostatic Unit of Quantity.**—If the unit of quantity of electricity is chosen so that two such units placed at a distance of one centimetre apart in air exert a mutual force equal to one dyne, $k = 1$ and equation (6) becomes—

$$F = \frac{q_1q_2}{Kr^2}$$

**9.** When $r$ is varied, an amount of work is done equal to—

$$\int_{r_1}^{r_2} \frac{q_1q_2}{Kr^2}dr \text{ or } \frac{q_1q_2}{K}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \quad \ldots \ldots \quad (7)$$

where $r_2$ and $r_1$ are the initial and final values of $r$. If $r_2$ is infinite, the work done is—

$$+\frac{q_1q_2}{Kr_1} \quad \ldots \ldots \ldots \ldots \quad (8)$$

This is the work done by, or against, the electric force, according as $q_1$ and $q_2$ have like or unlike signs, in bringing them to a distance $r_1$ from an infinite distance apart.

The unit of work—*the erg*—is the work done by a dyne when it moves its point of application over the distance of one centimetre.

If $q_2 = 1$, the work done is—

$$+\frac{q_1}{Kr_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and is called the **Potential** of $q_1$ at a distance $r_1$, and is usually denoted by the letter $V$.

In the same way the potential at a point distant $r_1$, $r_2$, $r_3$— from any system $q_1$, $q_2$, $q_3$—of electric charges in air is given by—

$$+V = \frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \ldots = \int \frac{q}{r} \quad . \quad . \quad . \quad (10)$$

When the electric force does work in removing a positive unit of electricity from a point to an infinite distance away, $V$ is reckoned positive; but when a positive unit of electricity is removed from the point to an infinite distance away work is done against the electric force by some external agent, $V$ is reckoned negative.

When a quantity of electricity is placed in an electric field, the force acting on it tends, therefore, to move it from places of high potential to places of low potential.

The surface of an insulated conductor has every point at the same potential, since, if such were not the case, any electricity on its surface would flow from places of high potential on its surface to places of low potential until equilibrium was established.

Such a surface is called an **Equipotential Surface,** and the value of the potential at any point of the surface is called the potential of the conductor. The potential of the earth is taken arbitrarily to be the zero of potential.

**10.** The potential of an insulated conductor depends not only on its size, form, and the quantity of electricity on its surface, but also upon its position relative to neighbouring conductors. An arrangement consisting of two conductors, one of which is insulated and the other (usually, but not necessarily) connected to the earth, is called a **Condenser**.

The quantity of electricity which must be given to the insulated conductor in order that the difference of the potentials of the two conductors (or coatings, as they are sometimes called) may be unity, is called the capacity of the condenser.

**Capacity.**—The capacity of a single conductor is the charge necessary to raise its potential from zero to unity.

The C.G.S. unity of capacity is possessed by a condenser whose

coatings have unit difference of potential when unit quantity of electricity is imparted to one of them.

The practical unit of capacity is called a **Farad**, and is $10^{-9}$ times the C.G.S. Electromagnetic Unit.

The difference of potential, $V$, of the coatings of a condenser is proportional to the total charge $Q$ of electricity; hence, if $C$ is the capacity of the condenser—

$$Q = CV \quad \ldots \ldots \quad (11)$$

It is to be noted that one of the coatings of the condenser may be the earth itself, in which case $V$ is the actual potential of the other coating.

### Work done in Charging a Condenser.

**11.** The work done in charging a condenser so that the difference of the potential of its coatings may be $V$ with a charge $Q$ is given by—

$$W = \int_0^Q VdQ$$

$$\text{by equation (11)} = \int_0^Q \frac{Q}{C}dQ$$

$$= \frac{1}{2}\frac{Q^2}{C}$$

$$= \tfrac{1}{2}CV^2 \quad \ldots \ldots \quad (12)$$

### PROBLEMS ON CHAPTER I.

1. Find the force in dynes due to two magnet poles of strengths 4 and 10 respectively, placed at a distance of 5 centimetres apart.

*Answer.* 1·6 dynes.

2. Find the equation to the line of force passing through a given point due to two equal and opposite magnet poles.

*Answer.* $\cos\theta_1 - \cos\theta_2 =$ constant, where $\theta_1$ and $\theta_2$ are the angles between the line joining the two poles, and the lines joining the respective poles to any point in a line of force.

3. Find the equation to the line of force passing through a given point due to two magnet poles of strength $m$ and $m'$ respectively.

*Answer.* Cos $\theta_1 + k \cos \theta_2$ = constant, where $k$ is the ratio of the poles' strengths; and $\theta_1$, $\theta_2$ have the same meanings as in Question 2.

4. What current must pass through a solenoid having ten turns per centimetre of length in order that the field strength in its interior may be $5\pi$ units?

*Answer.* 1·25 amperes.

5. Show that if two coils are so situated that all the lines of force due to a current passing through one coil interlinks the other, then the coefficient mutual induction of the two coils is the square root of the product of the coefficient of self-induction of the two coils.

6. What is the energy of charge of a condenser whose capacity is 5 microfarads when its potential is 1 volt? What is the charge on the condenser?

*Answers.* 25 ergs; $5 \times 10^{-7}$ coulombs.

7. What is the specific inductive capacity of the dielectric when the mutual force exerted by two quantities of electricity, each equal to 100 coulombs, situated 10 centimetres apart, is 0·5 dyne?

*Answer.* 2.

8. Show that lines of force always cut the equipotential surfaces at right angles.

9. Show that the self-induction of a coil is proportional to the square of its number of turns.

# CHAPTER II.

Induced Electromotive Forces—Faraday's Law—Induced Currents—Lenz's Law—Self and Mutual Induction—Energy of a Magnetic Field due to Electric Currents—Currents in Inductive Circuits.

## INDUCED ELECTROMOTIVE FORCES.

**12. Induced E.M.F.s.**—Faraday showed experimentally that if a conductor is moved in a magnetic field, or if there is by any means produced a relative motion between the conductor and the lines of magnetic force, so that the conductor cuts the lines of force, an electromotive force is induced in it, and also that the **rate of cutting** the lines of force is a measure of the E.M.F. induced. Thus, if $dN$ lines of force are cut in an infinitely small time $dt$ by a moving conducting wire, the E.M.F. $e$ generated is given by—

$$e = k . \frac{dN}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $k$ is some constant determined by the units employed. If, further, we take as the unit of E.M.F. that which is induced when one centimetre length of wire cuts one line of force per second, $k = -1$, and the above equation becomes—

$$e = -\frac{dN}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The negative sign is prefixed because the E.M.F. is always induced in such a direction as to oppose the cause which produces it.

Equation (2) simply means that the E.M.F. induced in the conductor is numerically equal, at each instant, to the rate of which it is cutting lines of magnetic force.

The C.G.S. unit of electromotive force upon which equation (2) is based is so small that for practical purposes it is convenient

to take as the unit of E.M.F. that which is induced in a conductor which cuts $10^8$ C.G.S. lines of force per second. This practical unit is called a **Volt.**

The direction of the induced E.M.F. depends not only on the direction of motion of the conductor, but also on the direction of the magnetic field. The following is an easily remembered rule for finding the direction of the E.M.F. when the directions of motion and of the field are given.

Suppose that the directions of motion and of the field are given by the lines *OM*, *OF* respectively (Fig. 1), then the induced E.M.F. is given in direction by the line *OE* drawn at right angles to both *OM* and *OF* in such a way that beginning at *E* and going round in a counter-clockwise direction, the order is *EMF*.

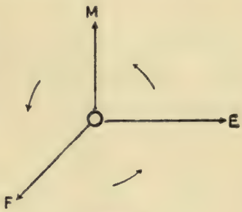

FIG. 1.

## INDUCED CURRENTS.

**13.** If the conductor which is moving in a magnetic field forms a closed circuit, the induced E.M.F. will produce an electric current, and, neglecting self-induction, the value of the current at any instant is given by—

$$i = -\frac{\frac{dN}{dt}}{r}$$

where $r$ is the total resistance of the circuit, and $N$ is the number of magnetic lines threading through the circuit at that instant. Here we suppose that the current $i$ is due solely to the induced E.M.F.

**E.M.F. of Self-induction.**—If a closed circuit carries a current $i$ the number of lines of force of self-induction is $Li$ (§ 6). If, by any means, $i$ is varied, an E.M.F. equal to $-\frac{d(Li)}{dt}$ is induced, or, if $L$ is considered constant.

$$e = -L\frac{di}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is called the E.M.F. of self-induction, and is that E.M.F. which is induced in a conductor when the magnetic field due to the current flowing in it varies.

**E.M.F. of Mutual Induction.**—If two closed circuits $A$ and $B$, carry respectively currents $i_1$ and $i_2$, the number of lines of force linking $A$ due to mutual induction is $Mi_2$ (see § 7, Chap. I.), and the number linking $B$ due to mutual induction is $Mi_1$. If the two currents vary there will be E.M.F.s due to mutual induction in $A$ and $B$ respectively equal to

$$-\frac{d(Mi_2)}{dt} \text{ and } -\frac{d(Mi_1)}{dt}$$

or if the coefficient of mutual induction, $M$, is taken to be constant, these become respectively—

$$-M\frac{di_2}{dt} \text{ and } -M\frac{di_1}{dt} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which mean that the induced E.M.F. of mutual induction in either coil is numerically equal to the rate at which that coil cuts the lines of magnetic force due to the current flowing in the other.

## Energy of a Magnetic Field due to Electric Currents.

**14.** Case I. Field due to a **single electric circuit.**

If a conducting wire carries an electric current, a magnetic field is produced. Suppose that at that time $t$ after the circuit is made the value of the electric current is $i$ and that the coefficient of self-induction of the circuit is $L$.

The magnitude of the E.M.F. which opposes the growth of the current is—

$$L\frac{di}{dt}, \text{ by § 13, equation (3)}$$

and the rate at which work is being done which is the product of the corresponding instantaneous values of the current and E.M.F. is—

$$iL\frac{di}{dt}$$

If $I$ is the maximum value of the current (when the steady state is attained) the total work done against the counter E.M.F. is the sum of the products—

$$iL\frac{di}{dt}\delta t$$

where $\delta t$ is a small element of time corresponding to the value $i$

of the current. Thus the work, $W$, expended in creating the magnetic field is given by—

$$W = \int_0^I iL\frac{di}{dt}dt$$
$$= \tfrac{1}{2}LI^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

**Energy of the Magnetic Field.**—This is the energy expended in driving the current against the counter E.M.F. of self-induction from the instant at which the circuit is made to the time when the current attains its maximum value, and it has its equivalent in the potential energy stored up in the magnetic field.

CASE II. Field due to currents in **two mutually inductive circuits.**

If $I_1$ and $I_2$ are the maxima values of the two currents in the two circuits respectively, the energy expended in driving the currents against their respective E.M.F.s of self-induction will (by Case I.) be—

$$\tfrac{1}{2}L_1I_1^2 \text{ and } \tfrac{1}{2}L_2I_2^2$$

where $L_1$ and $L_2$ are their respective coefficients of self-induction.

There will, however, be opposing E.M.F.s in each circuit due to mutual induction. If $M$ be the coefficient of mutual induction of the two circuits, and $i_1$ and $i_2$ be the instantaneous values of their respective currents, the opposing E.M.F. in circuit 1 due to mutual induction is (§ 13)—

$$M\frac{di_2}{dt}$$

and that in circuit 2 is—

$$M\frac{di_1}{dt}$$

The rate at which work is being done against mutual induction in the two circuits taken together is therefore—

$$i_1M\frac{di_2}{dt} + i_2M\frac{di_1}{dt}$$

and the whole energy expended in driving the currents against

their respective E.M.F.s of mutual induction while the currents rise from $o$ to $I_1$ and $o$ to $I_2$ respectively is given by—

$$\begin{aligned} W' &= \int\left(i_1 M\frac{di_2}{dt} + i_2 M\frac{di_1}{dt}\right)dt \\ &= M\int\left(i_1\frac{di_2}{dt} + i_2\frac{di_1}{dt}\right)dt \\ &= M\int d(i_1 i_2) = MI_1 I_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6) \end{aligned}$$

This together with the work done in driving the currents against their respective E.M.F.s of self-induction is the total energy expended in creating the magnetic field, and the total energy of the magnetic field is therefore given by—

$$W = \tfrac{1}{2}L_1 I_1^2 + \tfrac{1}{2}L_2 I_2^2 + MI_1 I_2 \quad . \quad . \quad . \quad . \quad (7)$$

This energy is stored up in the magnetic field and is restored to the circuits when the currents are stopped.

### Electric Currents in Inductive Circuits.

**15.** Case I. **Electric Current in a Single Inductive Circuit.**—Let the resistance of the complete circuit be $r$ and its coefficient of self-induction $L$, and let a potential difference $e$ be applied between the terminals of the circuit.

If $i$ is the instantaneous value of the current, the instantaneous value of the E.M.F. necessary to drive it against the resistance of the circuit alone is, by Ohm's law—

$$ri$$

The instantaneous E.M.F. due to self-induction, and which opposes the passage of the current, is—

$$-L\frac{di}{dt}$$

The applied potential difference has, therefore, to balance the E.M.F. of self-induction by providing a component equal to $+L\frac{di}{dt}$, and also to provide a component equal $ri$ to drive the current against the ohmic resistance of the circuit.

We thus arrive at the equation—

$$L\frac{di}{dt} + ri = e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This is an equation between the E.M.F.s acting in the circuit at the instant at which the current is $i$, and if solved will give the value of $i$ in terms of $e$, $t$, and the constants $L$ and $r$ of the circuit. Since $e$ is usually capable of being expressed as a function of $t$, the above equation gives the value of the current at every instant of time.

We dwell at length on this equation, because it is of fundamental importance in the theory of alternating currents, and should be thoroughly understood.

A solution is not possible until $e$ is known in terms of $t$, and this is left for consideration in a later chapter. As the equation is of paramount importance, we proceed to consider it from a different standpoint, and deduce it from the law of **Conservation of Energy.**

The rate at which energy is being supplied to the circuit at the instant at which the value of the current is $i$ is the product

$$ei,$$

and this must equal the rate at which energy is being dissipated in heating the circuit together with that used in creating the negative field, that is by § 14, must equal—

$$ri^2 + iL\frac{di}{dt}$$

Thus we arrive at the equation—

$$iL\frac{di}{dt} + ri^2 = ei$$

or, dividing by $i$;

$$L\frac{di}{dt} + ri = e$$

It must be remembered that this equation is a relation between the impressed potential difference, the E.M.F. of self-induction and the E.M.F. necessary to drive the current against the resistance of the circuit *at that instant at which the value of the current is "i."*

CASE II. **Electric Currents in Two Mutually Inductive Circuits.**—Let $L_1$, $L_2$ be the coefficients of self-induction of the two circuits, $r_1$, $r_2$ their resistances, and $M$ their coefficient of mutual induction, and let potential differences whose

instantaneous values are $e_1$, $e_2$ be applied between their respective terminals. Let also $i_1$, $i_2$ be the corresponding instantaneous values of the currents flowing in the two circuits.

First consider one of the circuits alone.

The function of $e_1$ is to drive the current $i_1$ against the resistance $r_1$ of the circuit, and to balance the E.M.F.s due to self and mutual induction.

At any instant, therefore, $e_1$ must equal the sum of—

$$r_1 i_1,\ L_1 \frac{di_1}{dt},\ \text{and}\ M\frac{di_2}{dt}$$

thus we get the equation—

$$L_1\frac{di_1}{dt} + M\frac{di_2}{dt} + r_1 i_1 = e_1 \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In exactly the same way for the other circuit—

$$L_2\frac{di_2}{dt} + M\frac{di_2}{dt} + r_2 i_2 = e_2 \quad . \quad . \quad . \quad . \quad (10)$$

These are relations between the E.M.F.s in the two circuits respectively, and are also two simultaneous equations to determine $i_1$ and $i_2$ in terms of the other quantities.

The case of greatest practical interest is that in which $e_2 = o$. The equations then refer to the primary and secondary circuits of a transformer or an induction coil, and are discussed fully in the chapter on Transformers.

Equations (8), (9), and (10) are formed on the supposition that the coefficients of self and mutual induction are constant. This condition will often hold good since, in most alternating-current machinery, to which these equations can refer, the induction in the iron does not approach saturation.

Equations (9) and (10) may be deduced from the law of conservation of energy in the same way as equation (8); but this is left as an exercise for the reader.

## PROBLEMS ON CHAPTER II.

1. A straight conductor 1 metre long is displaced parallel to itself and at right angles to a uniform field at a rate of 100 metres per second. If the field strength is 100 C.G.S. units, what E.M.F. is generated in the conductor?

*Answer.* 1 volt.

2. A closed circuit whose coefficient of self-induction is 0·5 henry carries a steady current of 100 amperes: what is the energy stored up in the magnetic field due to the current?

*Answer.* $0·25 \times 10^{11}$ ergs.

3. Two closed circuits, whose coefficients of self-induction are respectively 0·5 and 0·75 henry, and whose coefficient of mutual induction is 0·6 henry, carry steady currents of 10 and 20 amperes respectively: what is the energy of the magnetic field due to the currents?

*Answer.* $2·95 \times 10^9$ ergs.

4. What becomes of the energy of a magnetic field when the current which creates it is interrupted?

5. The axle of a railway carriage wheel, 5 feet 3 inches long, moves at the rate of 60 miles an hour in the earth's vertical field [= 0·47 C.G.S. units]. What E.M.F. is induced in it?

*Answer.* 0·002 volt, nearly.

# CHAPTER III.

The Production of Alternating Electromotive Forces—Generators—Armature Reaction—Circuit with Constant Inductance—Impedance.

## THE PRODUCTION OF ALTERNATING ELECTROMOTIVE FORCES.—GENERATORS.

**16.** When a closed circuit moves in a magnetic field the rate of cutting lines of magnetic force is the same as the rate of change of the magnetic flux through the circuit, for since the lines of force themselves always form closed curves the flux through the circuit cannot change without a corresponding cutting of magnetic lines.

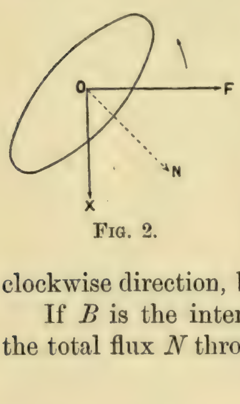

FIG. 2.

Suppose that a single plane closed circuit rotates with uniform angular velocity about an axis in its own plane at right angles to the direction of a uniform magnetic field.

Let $O$ be a point of the axis $OX$ of rotation of the coil, $OF$ the direction of the field at right angles to $OX$, and $ON$ the normal to the plane of the coil at time $t$, and let the angle, measured in a counter-clockwise direction, between $OF$ and $ON$ be $\theta$.

If $B$ is the intensity of the field, and $A$ the area of the coil, the total flux $N$ through the circuit at the time $t$ is given by—

$$N = AB \cos \theta$$

since $A \cos \theta$ is the projection of $A$ at right angles to $B$, or $B \cos \theta$ is the component of the intensity normal to the plane of $A$.

If the angular velocity of the coil is $p$, and $t$ is measured from the instant when $ON$ and $OF$ coincide, then—

$$\theta = pt$$

so that—

$$N = AB \cos pt$$

The E.M.F. induced in the moving circuit is, therefore, given by—

$$e = -\frac{dN}{dt}$$
$$= ABp \sin pt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is the rate of change of the magnetic flux through the circuit.

The value of $e$ is therefore zero when $pt$ is 0 or any multiple of $2\pi$; it then becomes positive and attains a maximum value $ABp$ when $pt$ is $\frac{\pi}{2}$, or $\frac{\pi}{2}$ + any multiple of $2\pi$; it becomes zero when $pt = \pi$, or any odd multiple of $\pi$, and then becomes negative, and is a negative maximum, and equal to $-ABp$ when—

$$pt = \frac{3\pi}{2} + \text{any multiple of } 2\pi$$

As $\theta$ increases from 0 to $2\pi$ the coil rotates through one complete revolution and the value of $e$ goes through a complete cycle of changes which repeats itself in every succeeding revolution.

If the coil consists of $n$ complete turns each having an area

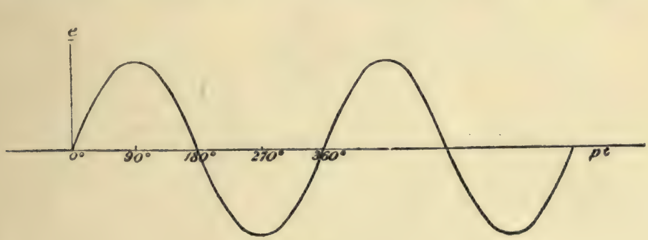


FIG. 3.

equal to $A$, and arranged so that all the turns are placed in series and form one circuit of $n$ loops, the induced E.M.F. is increased $n$-fold, and is given by—

$$e = nABp \sin pt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If $E$ is the maximum value of $e$—

$$E = nABp \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and—

$$e = E \sin pt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Such an E.M.F. is called **Alternating** because during each revolution of the coil it rises from the value zero to a positive maximum, then passes through the value zero and becomes negative, attaining a negative maximum numerically equal to its positive one, and then rises to the value zero again.

Fig. 3 is a graphic representation of equation (4).

If the extremities of the revolving coil are connected to two collecting rings mounted on the axis of rotation and insulated from each other so that the coil can be revolved between the poles of an electro-magnet, we have an **alternating-current generator,** or an **alternator.**

An alternating current may be supplied to an external circuit by means of two brushes making rubbing contacts with the collecting rings.

The simplest form of alternator consists of either a drum or a ring armature which rotates between the poles of separately excited field magnets. A diagrammatic representation is given in Fig. 4.

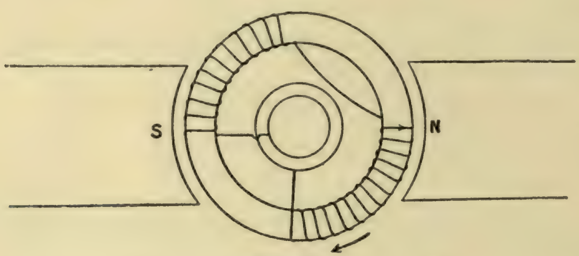


Fig. 4.

In such a two-pole machine the E.M.F. developed passes through a complete cycle of changes or alternation once every complete revolution of the revolving part.

As it is necessary for such purposes as electric lighting to have not less than about forty complete alternations per second, a two-pole alternator would be impracticable except in very small machines. If, instead of having but two poles, four alternately

north and south poles are placed 90° apart, there will be two complete alternations per revolution of the armature. If six alternately north and south poles are placed 60° apart, there will be three complete alternations per revolution, and so on. Machines having more than two poles are called multipolar alternators. Except for special, or experimental, purposes, multipolar alternators are universally used for commercial work. Fig. 5 gives a diagrammatic representation of a 6-pole alternator. We do not propose to enter into an exhaustive description of the various types of alternators, as such would be beyond the scope of this treatise. We will, however, give a general idea of the various types of machines met with in practice.

**17. Types of Alternators.**—There are, broadly speaking, three types of alternators: (1) those in which separately excited poles—or **field magnets**—are stationary and inducing coils—or **armature**—rotate; (2) those in which the armature is stationary and the field magnets rotate; and (3) **Inductor Alternators,** or those in which both the field magnets and armature are stationary and the flux through the armature coils is varied by masses of iron rotating in the gap between the field coils and the armature coils.

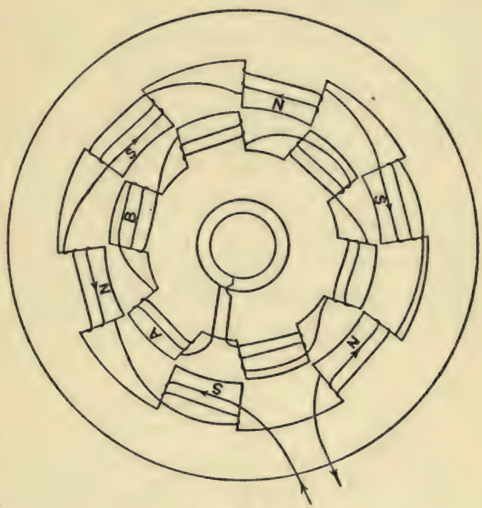

Fig. 5.

In alternators giving a single alternating E.M.F. the magnet poles occupy about one half the circumference of the machine, as shown in Fig. 5, as also do the coils on the armature. The reason for this is seen by reference to Fig. 5, in which the poles are the same width as the spaces between them. If the armature coil $A$ were of greater angular width than the distance between two consecutive poles $N$, $S$, it would begin cutting lines issuing

from the S. pole before it had finished cutting those issuing from the N. pole, or *vice versâ* according to the direction of rotation of the armature, with the result that there would be a differential effect and the E.M.F. developed would be less than it should be for the number of conductors on the armature.

Of the different types of alternators it is difficult to say which is the best. There is one consideration, however, which is worth while bearing in mind. Alternators are usually employed for generating high E.M.F.s, more especially where energy has to be transmitted for a considerable distance; E.M.F.s up to 20,000 volts being not uncommon. Such being the case, it is advisable that the armature coils in which the high pressure is generated, and which are subjected to great electrical strain, should not be subjected to mechanical strain. This consideration leads us to prefer those types in which the armature is stationary, so that where high pressures are to be generated machines of the stationary field type or inductor alternators are to be preferred.

We shall for the present assume that the E.M.F.s of all alternators are capable of representation as either a single sine function of the time, or as the sum of a series of sine and cosine functions of the time. In either case we can base our calculations on the representation of the E.M.F. as a single sine function (see Chap. IX.).

## Armature Reaction.

**18.** The E.M.F. generated in the armature of an alternator is due to a combination of two distinct causes. First, an E.M.F. is generated by the rotation of the armature in a magnetic field, and, secondly, there is an E.M.F. set up in the armature by the variations of the armature current itself. This latter may be called the E.M.F. of self-induction of the armature. Its effect is to change the intensity and the direction of the resultant magnetic field through which the armature rotates. It is sometimes termed **Armature Reaction.** We shall show in a later chapter (see Chap. XIII.) that the field strength of an alternator may, according to circumstances, be either increased or diminished by armature reaction.

The number of lines of force passing through the armature due to its own current $i$ is $Li$, where $L$ is its coefficient of self-

induction. The value of the armature reaction at any instant is therefore, by § 13, Chap. II., equal to—

$$-\frac{d(Li)}{dt}$$

This is variable, since $i$ is a function of the time, and the value of $L$ depends upon the variable position of the armature coils relative to the field poles, and also upon the induction in the iron of both armature and field magnets.

There is also another source of variation of the magnetic field which should be included under the head of armature reaction. The variation of the flux produced by the armature currents will induce a variable current in the field-magnet coils, and a consequent variation of the flux due to the field coils themselves. This variation is proportional to the mutual induction between the armature coils and field coils. Under the same heading may be included any variation in the strength of the field due to eddy currents induced in the field magnets.

We shall not, however, for the present, inquire further into the effect of armature reaction upon the E.M.F. of an alternator, but shall content ourselves with the assumption that both the E.M.F. of the alternator and the potential difference between its terminals are capable of representation by means of sine (or cosine) functions of the time.

**19.** We now proceed to obtain the relation between the current, the E.M.F., and the constants of a circuit in which we can treat the self-induction as constant.

The product of the coefficient of self-induction and $2\pi$ times the number of complete cycles per second is called the **Reactance** of a circuit. An extended definition of reactance is given in § 45.

The number of complete alternations per second is called the **Frequency,** and is usually denoted by the letter $n$.

The time taken for one complete cycle is called the **Periodic Time.**

We shall denote the whole electro-motive force round a circuit by the letters E.M.F., and the potential difference between any two points in a circuit by the letters P.D.

### Current in a Circuit of Constant Inductance with a Constant Potential Difference between its Terminals.

**20.** Let $L$ be the coefficient of self-inductance of the circuit,
$r$ ,, resistance of the circuit,
$e$ ,, constant P.D.,
$i$ ,, current at any instant,
$t$ ,, time from the instant when the current is made.

The equation from which to determine the current is (see § 15, Chap. II.)—

$$L\frac{di}{dt} + ri = e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The complete solution of this equation is (see Appendix)—

$$i = \frac{e}{r}\left(1 - \varepsilon^{-\frac{rt}{L}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $\varepsilon$ is the base of Naperian logarithms, and equals 2·7 approximately.

The exponential term occurring in the solution shows that the current does not theoretically attain the steady value $\frac{e}{r}$ for an infinite time; it, however, practically attains this value after a very short time.

The time $T$ taken for $i$ to reach the value $\frac{1}{\varepsilon} \cdot \frac{e}{r}$ is given by—

$$T = \frac{L}{r}\log_{\varepsilon}\left(\frac{\varepsilon}{\varepsilon - 1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$\frac{L}{r}$ is sometimes called the **Time Constant** of the circuit.

### Current in a Circuit of Constant Inductance with an Alternating Potential Difference between its Terminals.

**21.** Let $L$ be the coefficient of self-induction of the circuit,
$r$ ,, resistance of the circuit,
$n$ ,, frequency ,, ,,
$e = E \sin pt$ be the applied P.D.,
$p = 2\pi n$,
$i$ be the current in the circuit at any instant.

The equation from which to determine the current is then, by §§ 15 and 16—

$$L\frac{di}{dt} + ri = E \sin pt \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The complete solution of this equation is (see Appendix)—

$$i = \frac{E \sin (pt - \theta)}{\sqrt{r^2 + p^2L^2}} + A\varepsilon^{-\frac{rt}{L}} \quad . \quad . \quad . \quad . \quad (9)$$

where $\theta$ given by $\tan \theta = \frac{pL}{r}$ . . . . . . . . . . . . (10)

$A$ is a constant
and $\varepsilon$ is the base of Naperian logarithms, and equals 2·7 nearly.

After a very short time $\varepsilon^{\frac{-rt}{L}}$ becomes negligibly small, and the current attains a steady periodic state represented by the equation—

$$i = \frac{E \sin (pt - \theta)}{\sqrt{r^2 + p^2L^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If we write $I$ for $\frac{E}{\sqrt{r^2 + p^2L^2}}$, equation (10) takes the form—

$$i = I \sin (pt - \theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$I$ is therefore the maximum value of the current.

Equation (8) shows that the P.D. is zero when $t = 0$, and equation (12) shows that the resulting current is zero, and increasing in the same direction when—

$$t = \frac{\theta}{p},$$

that is, the current is a sine function of the time, and has its zero and maximum values later in point of time than, or lags behind, the P.D. by an amount $\frac{\theta}{p}$.

The E.M.F. of self-induction is given by—

$$- L\frac{di}{dt} = - pLI \cos (pt - \theta)$$
$$= pLI \sin \left(pt - \theta - \frac{\pi}{2}\right)$$

which shows that the E.M.F. of self-induction is also a sine

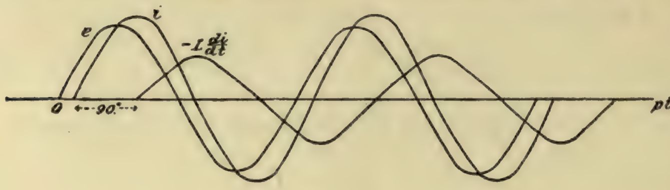


FIG. 6.

function of the time, and lags behind the current in time by an amount given by—

$$t = \frac{\pi}{2p}$$

Now, since—

$$\sin(pt - \theta) = \sin(pt - \theta - 2\pi)$$

the **Periodic Time** is given by—

$$t = \frac{2\pi}{p}$$

therefore **the E.M.F. of self-induction lags behind the current by a quarter of a period.**

The P.D., current, and E.M.F. of self-induction are graphically represented in Fig. 6.

We have seen that the maximum value of the current in an inductive circuit is given by—

$$I = \frac{E}{\sqrt{r^2 + p^2L^2}}$$

The quantity $\sqrt{r^2 + p^2L^2}$ is called the **Impedance** of the circuit.

In a non-inductive circuit we have—

$$I = \frac{E}{resistance} \qquad \text{(Ohm's law)}$$

and in an inductive circuit—

$$I = \frac{E}{impedance}$$

The value of $pt + \theta$ at any instant is called the **Phase** of

any current, $i = I \sin pt + \theta$ (or E.M.F.) at that instant, since it measures the displacement from its zero value.

Two alternating currents $i_1 = I_1 \sin pt$ and $i_2 = I_2 \sin(pt - \theta)$ have phases differing by an amount (or angle) $\theta$, which is called their **Phase Difference**.

Equation (10) shows that in an inductive circuit the difference of phase between the current and P.D. is given by—

$$\tan \theta = \frac{pL}{r}$$

## PROBLEMS ON CHAPTER III.

1. A straight conductor 1 metre long moves parallel to itself in simple periodic motion of amplitude 1 centimetre at right angles to a uniform field of strength 1000 C.G.S. units. If the periodic time is 0·001 second, what is the maximum E.M.F. generated in the conductor?

*Answer.* 6·28 volts.

2. A circular conductor whose diameter is 1 metre revolves about a diameter with an angular velocity of 10 revolutions per second in a uniform field of strength 10,000 C.G.S. units, so that in one position of the conductor the lines of force pass normally through it: what is the maximum E.M.F. developed?

*Answer.* 493·5 volts.

3. A steady E.M.F. is applied between the terminals of a conductor whose self-induction is 0·25 henry, and resistance 1 ohm: how long will it take the current to reach one-tenth its steady value?

*Answer.* 0·025 second.

4. What is the maximum value of the current in a conductor whose self-induction is 0·5 henry, and resistance 2 ohms, when an alternating E.M.F., whose maximum value is 100 volts, and frequency 100 cycles per second, is applied between its terminals?

*Answer.* 0·3176 ampere.

5. What is the impedance of the circuit in Question 4?

*Answer.* 314·2 ohms.

6. What is the difference in phase between the E.M.F. and current in Question 4?

*Answer.* 86° 20′.

7. An alternator gives a potential difference between its brushes of 1000 volts when the current is 100 amperes; if the current is increased to 150 amperes, the external circuit being non-inductive, the potential difference falls to 920 volts. What is the E.M.F. developed in the armature, the frequency being constant?

*Answer.* 1160 volts.

8. What is the impedance of the armature in Question 7?

*Answer.* 1·6 ohms.

9. What would be the potential difference between the brushes of the machine in Question 7 if the current was increased to 175 amperes?

*Answer.* 880 volts.

# CHAPTER IV.

Capacity in Alternating Current Circuits—Combination of Capacity and Inductance in series—Combination of Capacity and Inductance in Parallel—Reactance—Resonance—Root Mean Square Values.

## CAPACITY IN ALTERNATING CURRENT CIRCUITS.

**22.** If a condenser is placed in an alternating current circuit, an alternating current will pass to an extent depending upon the capacity of the condenser and the magnitude and frequency of the P.D. applied between the terminals of the condenser. The resulting current is the charge and discharge current as the P.D. between the condenser terminals alternates.

If $Q$ is the quantity of electricity on the condenser at any instant, the current $i$ is the rate of variation of $Q$ and is given by—

$$i = \frac{dQ}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now, by § 10, Chap. I., if $C$ is the capacity of the condenser and $V$ the P.D. between its terminals, we have the relation—

$$Q = CV$$

therefore—

$$\begin{aligned} i &= \frac{dQ}{dt} \\ &= C\frac{dV}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2) \end{aligned}$$

since $C$ is constant.

Now, suppose that the P.D. is given by—

$$V = V_0 \sin pt$$

Then—

$$\begin{aligned} i &= C\frac{dV}{dt} \\ &= CV_0 p \cos pt \\ &= CV_0 p \sin\left(pt + \frac{\pi}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad (3) \end{aligned}$$

which shows that **the current is in advance of the P.D. by a time given by**—

$$t=\frac{\pi}{2p}$$

**that is, by a quarter of a period.**

Thus, by reference to § 21, Chap. III., we see that whereas self-induction causes the current to lag behind the applied P.D., capacity causes the current to lead before the P.D., and whereas E.M.F. of self-induction lags a quarter of a period behind the current, the E.M.F. due to capacity leads a quarter of a period before the current. The E.M.F.s due to self-induction and capacity in the same circuit are thus seen to be in direct opposition, and may, under suitable conditions, completely neutralize one another.

We will now proceed to calculate the value of the current in alternating current circuits containing capacity.

**23. Electric Current in a Circuit containing Resistance and Capacity only.**—Consider a circuit of total resistance $r$ containing a condenser of capacity $C$. Let an alternating E.M.F. $e=E\sin pt$ be applied between the ends of the circuit, and let $i$ be the resulting current.

The function of $e$ is to drive the current $i$ against the resistance $r$ of the circuit, and to balance the E.M.F. $\frac{Q}{C}$ due to capacity, where $Q$ is the charge on the condenser at any instant. We, therefore, arrive at the equation—

$$ri+\frac{Q}{C}=E\sin pt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Differentiating this with respect to $t$, we get—

$$r\frac{di}{dt}+\frac{1dQ}{Cdt}=pE\cos pt$$

or—

$$r\frac{di}{dt}+\frac{i}{C}=pE\cos pt \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The complete solution of this equation is (see Appendix)—

$$i=\frac{E\sin(pt+\phi)}{\sqrt{r^2+\frac{1}{p^2C^2}}}+A\varepsilon^{-\frac{t}{Cr}} \quad . \quad . \quad . \quad . \quad (6)$$

where—

$$\tan\phi = \frac{1}{pCr}$$

$\varepsilon$ is the base of Naperian logarithms, and $A$ is a constant.

The exponential term in (6) soon dies out, and the current soon assumes the steady periodic state given by the equation—

$$i = \frac{E\sin(pt+\phi)}{\sqrt{r^2+\frac{1}{p^2C^2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This shows that the current leads before the E.M.F. by a time $t=\frac{\phi}{p}$. An example of such current and E.M.F. curves is given in Fig. 7.

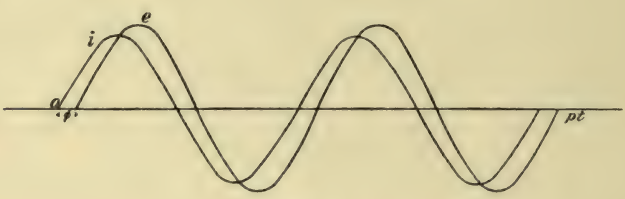


FIG. 7.

**24. Electric Current in a Circuit containing Resistance, Capacity, and Self-induction in Series.**—Let the total resistance of the circuit be $r$, the coefficient of self-induction $L$, the capacity $C$, the E.M.F. $e=E\sin pt$, and the resulting instantaneous value of the current $i$.

The E.M.F. $e$ has to drive the current $i$ against the resistance $r$, and to balance the E.M.F.s due to self-induction and capacity. Therefore—

$$L\frac{di}{dt}+ri+\frac{Q}{C}=E\sin pt \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where $Q$ is the quantity of electricity on the condenser at any instant.

Differentiating equation (8) with respect to $t$, we get—

$$L\frac{d^2i}{dt^2}+r\frac{di}{dt}+\frac{1}{C}\frac{dQ}{dt}=pE\cos pt$$

or by (1)—

$$L\frac{d^2i}{dt^2}+r\frac{di}{dt}+\frac{i}{C}=pE\cos pt \quad . \quad . \quad . \quad . \quad . \quad (9)$$

From this we find that when the current has reached a steady periodic state its value is given by (see Appendix)—

$$i=\frac{E\sin(pt-\phi)}{\sqrt{\left\{r^2+p^2\left(L-\frac{1}{p^2C}\right)^2\right\}}} \quad . \quad . \quad . \quad . \quad (10)$$

where—

$$\tan\phi=\frac{pL-\frac{1}{pC}}{r}$$

This shows that the current lags behind or leads before the E.M.F., according as $pL$ is greater or less than $\frac{1}{pC}$. If $pL=\frac{1}{pC}$ the self-induction and capacity exactly balance each other.

The quantity $pL-\frac{1}{pC}$ is called the **Reactance** of the circuit. In an inductive circuit the reactance is positive; in a circuit containing capacity only it is negative; in a circuit containing both inductance and capacity it is positive or negative according as $pL$ is greater or less than $\frac{1}{pC}$

When a circuit contains both inductance and capacity in such proportion that the reactance is zero, we have the phenomenon of electrical **Resonance.**

**25. Electric Currents in Parallel Circuits, one Branch of which contains Resistance and Self-induction and the other Resistance and Capacity.** —Let $r_1$ and $L$ be the resistance and self-induction in one branch, and $r_2$ and $C$ the resistance and capacity in the other branch. Let $i_1$, $i_2$ be the currents in the two branches and $v=V\sin pt$ the common P.D. between their terminals.

The equation for the inductive branch is—

$$L\frac{di}{dt}+r_1i_1=V\sin pt$$

and that for the capacity branch is—

$$r_2i_2+\frac{Q}{C}=V\sin pt \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where $Q$ is the quantity of electricity on the condenser at any instant. Therefore by §§ 21 and 23—

$$i_1 = \frac{V \sin (pt - \theta)}{\sqrt{r_1^2 + p^2L^2}} \quad . \quad . \quad . \quad . \quad (12)$$

$$i_2 = \frac{V \sin (pt + \phi)}{\sqrt{r_2^2 + \dfrac{1}{p^2C^2}}} \quad . \quad . \quad . \quad . \quad (13)$$

$$\text{where } \tan \theta = \frac{pL}{r_1}$$

$$\text{and } \tan \phi = \frac{1}{pCr_2}$$

Equations (12) and (13) show that for high frequencies (*i.e.* $p$ is large) nearly the whole of the current goes through the condenser branch.

The current $i$ in the main circuit is the resultant of the currents $i_1$ and $i_2$, and is given by—

$$i^2 = i_1^2 + i_2^2 + 2i_1i_2 \cos (\theta + \phi)$$

$$= i_1^2 + i_2^2 + 2i_1i_2\left\{\frac{pCr_1r_2 - pL}{\sqrt{r_1^2 + p^2L^2}\sqrt{1 + p^2C^2r_2^2}}\right\} \quad . \quad . \quad (14)$$

If—

$$L = Cr_1r_2$$

$$i^2 = i_1^2 + i_2^2$$

and $i_1$, $i_2$ differ in phase by a right angle.

## Measurements of Alternating Currents and Electromotive Forces.

**26.** In taking electrical measurements in alternating current circuits special types of instruments are necessary.

Instruments which can be used for alternating current measurements can usually be employed for direct current measurements, but the converse is not true.

All instruments for the measurement of alternating currents and E.M.F.s depend upon a **square law** and the deflections are proportional to the **mean square** of the quantity to be measured. Amongst instruments for the measurement of alternating currents may be mentioned **electro-dynamometers, and Kelvin**

**Balances;** amongst those for the measurements of E.M.F.s are **Electrostatic** and **Hot Wire Voltmeters.**

It is of importance, therefore, to clearly understand what it is that is actually measured by alternating-current instruments.

It has been stated that the deflections are proportional to the mean square of the quantity to be measured.

The instruments may be provided with a uniform scale and the square root of the reading taken as a measure of the current or E.M.F., or they may be graduated to read direct, in which case the scale will not be uniform. Whatever way the instrument is graduated, the quantity measured is, therefore, a **root mean square value.**

### Root Mean Square Values.

**27.** The reason why root mean square values are taken in alternating-current measurements is obvious, for since the average value of a periodic current is zero, an instrument in which the deflection is proportional to the current would show no deflection, and it is necessary to employ instruments which are deflected in the same direction whatever be the direction of the current through them, that is, we must employ instruments which measure the mean square of the periodic quantity

$$i = I \sin pt$$

where $i$ is the instantaneous value of the current at any time $t$; $p = 2\pi n$, $n$ being the frequency; and $I$ is the maximum value of the current.

The root mean square value of the current is obtained by dividing half the periodic time into an infinite number of parts, taking the sum of the squared values of the current at each of these points, dividing this sum by half the periodic time, and extracting the square root of the result.

The periodic time is $\frac{2\pi}{p}$, so that half the periodic time is $\frac{\pi}{p}$

If we write $A$ for the root mean square current, we have, therefore—

$$A = \sqrt{\frac{p}{\pi}\int_0^{\frac{\pi}{p}} I^2 \sin^2 pt\, dt}$$

$$= \frac{I}{\sqrt{2}} \quad \dots\dots\dots\dots \quad (15)$$

D

Or, the root mean square current is numerically equal to the **maximum value divided by** $\sqrt{2}$

If the E.M.F. is represented by—

$$e = E \sin pt$$

the root mean square E.M.F. $V$, is similarly given by—

$$V = \sqrt{\frac{p}{\pi}\int_0^{\frac{\pi}{p}} E^2 \sin^2 pt\, dt}$$
$$= \frac{E}{\sqrt{2}} \quad \ldots \ldots \ldots \ldots \quad (16)$$

We shall in future use the letters R.M.S. to indicate root mean square values.

It was proved, see § 24, that in a reactive circuit of self-induction $L$ and capacity $C$—

$$I = \frac{E}{\sqrt{\left\{r^2 + \left(pL - \frac{1}{pC}\right)^2\right\}}}$$

Therefore—

$$\frac{I}{\sqrt{2}} = \frac{\frac{E}{\sqrt{2}}}{\sqrt{\left\{r^2 + \left(pL - \frac{1}{pC}\right)^2\right\}}}$$

or—

$$A = \frac{V}{\sqrt{\left\{r^2 + \left(pL - \frac{1}{pC}\right)^2\right\}}} \quad \ldots \quad (17)$$

We see, therefore, that the equation—

$$\text{Current} = \frac{\text{E.M.F.}}{\textit{impedance}}$$

is true both for maximum and R.M.S. values of current and E.M.F.

## PROBLEMS ON CHAPTER IV.

1. What is the maximum current in a circuit having a capacity of 2 microfarads, and a resistance of 10 ohms, when an alternating E.M.F., whose maximum

value is 100 volts, and frequency 100 cycles, per second, is applied between its terminals?

*Answer.* 0·126 ampere.

2. What is the impedance of the circuit in Question 1?

*Answer.* 795·2 ohms.

3. What is the difference in phase between E.M.F. and current in Question 1? Draw curves respecting them.

*Answer.* 82° 50′.

4. What is the self-induction of a coil through which, when 100 R.M.S. volts is applied between its terminals, a current of 5 amperes passes, the frequency being 80 periods per second, and the resistance of the coil being 0·5 ohm?

*Answer.* 0·0386 henry, nearly.

5. The maximum value of an alternating current is 100 amperes: what is its average value over half a period from zero to zero?

*Answer.* $\frac{200}{\pi}$ amperes.

6. The R.M.S. value of an alternating E.M.F. is 100 volts: what is its maximum value?

*Answer.* 141·4 volts.

7. A coil having a self-induction of 0·05 henry allows a current of 3 R.M.S. amperes to pass when the frequency is 100: what is the P.D. between its terminals, the resistance of the coil being neglected?

*Answer.* 94·25 volts.

8. What is the P.D. in Question 7, if the resistance of the coil is 5 ohms?

*Answer.* 95·4 volts.

9. What is the frequency when a current of 1 ampere is sent through a coil having a self-induction of 0·75 henry, a P.D. of 200 volts being applied between its terminals, the resistance of the coil being negligible?

*Answer.* 42·44.

10. What is the frequency in Question 9, if the resistance of the coil is 10 ohms?

*Answer.* 42·4.

11. A condenser of capacity 10 microfarads is connected between the terminals of an alternator giving 1000 volts at a frequency of 50 periods per second: what is the current?

*Answer.* $\pi$ amperes.

12. If a resistance of 10 ohms is inserted in series with the condenser in Question 11, what current will pass?

*Answer.* 3·14 amperes.

13. If a self-induction of 0·05 henry is placed in series with a capacity of 1 microfarad, and a P.D. of 100 volts, at a frequency of 100 periods per second, be applied between the extreme terminals, what is the current?

*Answer.* 0·0641 ampere.

14. A circuit contains in series a resistance of 10 ohms, a self-induction of 0·5 henry, and a capacity of 0·5 microfarad: what is the current if the P.D. between the terminals of the arrangement is 100 volts, and the frequency 80 periods per second?

*Answer.* 0·0268 ampere.

15. What is the fall of potential (1) along the resistance, (2) the self-induction, and (3) the capacity in Question 14?

*Answer.* (1) 0·268 volt, (2) 6·74 volts, (3) 1066·3 volts.

16. What value of the capacity will make the circuit in Question 14 non-reactive?

*Answer.* 7·9154 microfarads.

17. What is the difference in phase between E.M.F. and current in Question 14?

*Answer.* 89° 51′.

18. Prove that the R.M.S. value of a simple periodic function is the maximum value divided by $\sqrt{2}$, employing the simple trigonometrical method as on page 39.

# CHAPTER V.

Expression for Power—Measurement of Power.

## POWER GIVEN TO ALTERNATING-CURRENT CIRCUITS.

**28.** The power given to a circuit by a continuous current is the product of the current flowing in it and the P.D. between its terminals, and is given in watts when the current is given in amperes and the P.D. in volts. The power given to circuit by an alternating current cannot be determined in this fashion, for whereas in a continuous-current circuit the current and P.D. always act in the same direction round the circuit, in an alternating-current circuit there are, in general, times occurring periodically when the current and P.D. act in opposition and the circuit is giving back energy to the source.

We have seen (see Chap. IV., § 24) that when an alternating P.D. $e = E \sin pt$ is applied between the terminals of a circuit containing resistance, self-induction, and capacity in series, the resulting current is given by—

$$i = \frac{E \sin (pt - \theta)}{\sqrt{\left\{r^2 + \left(pL - \frac{1}{pC}\right)^2\right\}}}$$

$$= I \sin (pt - \theta)$$

$$\text{where } \tan \theta = \frac{pL - \frac{1}{pC}}{r}$$

the notation being the same as in Chapter IV.

The power being given to the circuit at any instant of time $t$ is the product of the P.D. and corresponding current at that instant, that is the product—

$$EI \sin pt \sin (pt - \theta)$$

The energy given to the circuit during a small interval of time $dt$ is therefore—

$$EI \sin pt \sin (pt - \theta)dt$$

If we divide a complete period of the current into an infinitely large number of infinitely small times $dt$, and take the sum of the energies given to the circuit during those times, we shall obtain the total energy given to the circuit during a time equal to a periodic time of the current, and if, further, we divide the expression thus obtained by the periodic time, we shall have the average or mean power given to the circuit.

Denoting the mean power by $P$, we therefore have—

$$\begin{aligned} P &= \frac{p}{2\pi}\int_0^{\frac{2\pi}{p}} EI \sin pt \sin (pt - \theta)dt \\ &= \frac{EI}{2} \cos \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1) \end{aligned}$$

That is **the power given to an alternating-current circuit is half the product of the maximum current and the maximum P.D. multiplied by the cosine of their phase difference.**

The expression for the mean power may be written—

$$\begin{aligned} P &= \frac{E}{\sqrt{2}} \cdot \frac{I}{\sqrt{2}} \cos \theta \\ &= VA \cos \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2) \end{aligned}$$

where $E$ and $I$ are the R.M.S. values of P.D. and current respectively.

**Thus the power given to an alternating-current circuit is the product of the R.M.S. values of the current and P.D. multiplied by the cosine of their phase difference.**

The R.M.S. values $E$ and $I$ are the quantities measured respectively by a voltmeter placed between the terminals of the circuit and an ammeter placed in the circuit. Since $\cos \theta$ is always less than unity, we see that the product of amperes and volts is, unless $\theta = 0$, greater than the power given to the circuit.

In contradistinction to the **true power,** $EI \cos \theta$, the product $EI$ is called the **apparent power,** and the ratio

of the true power to the apparent power is called the **power factor.** Thus—

$$\text{Power factor} = \frac{\text{true power}}{\text{apparent power}}$$
$$= \cos\theta$$

The expression for the mean power given to a circuit may be deduced in the following simple manner.

The instantaneous power given to the circuit is—

$$EI \sin pt \sin (pt - \theta)$$
$$= \frac{EI}{2} \cdot 2 \sin pt \sin (pt - \theta)$$
$$= \frac{EI}{2} \{\cos\theta - \cos(2pt - \theta)\} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now the average value of $\cos(2pt - \theta)$ taken over a complete period is zero, since for every positive value it has an equal negative value. Also $\cos\theta$ is constant; therefore the taking the average value of the power over a complete period, we get—

$$P = \frac{EI}{2}\{\text{average value of } \cos\theta - \text{average value of } \cos(2pt - \theta)\}$$
$$= \frac{EI}{2} \cos\theta$$
$$= EI \cos\theta$$

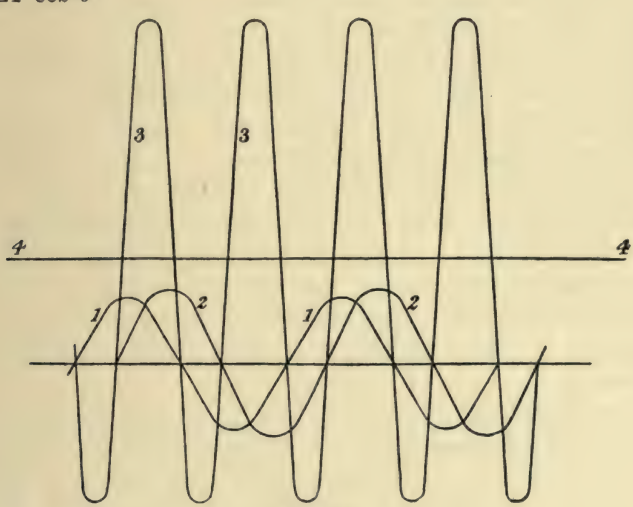


FIG. 8.

In Fig. 8 curve 1 is the curve of the P.D. Curve 2 shows the

resulting current in an inductive circuit. The ordinates of curve 3 are the products of corresponding ordinates of curves 1 and 2, and the curve shows the variation of the power. The straight line 4 in the figure shows the mean power. A close study of these curves will amply repay the reader.

## Measurement of Power.

**29.** Since the power given to an alternating-current circuit cannot be determined by the use of a single ammeter and voltmeter, special means for its measurement have to be devised.

Various methods involving combinations of ammeters and voltmeters may be employed, but the simplest method of measuring power is the use of a single instrument called a **Wattmeter.**

The wattmeter generally consists of two vertical coils of wire arranged so that one is fixed and the other is capable of rotation on a vertical axis. The two coils are placed so that their planes are at right angles to each other. One of the coils, a thick coil of few turns, is placed in series with the circuit in which the power has to be measured, while the other coil, a long thin wire coil of high resistance, is placed directly across the terminals of the circuit. The thin wire coil has so high a resistance that the current passing through it is very small, and should be arranged so as to have as low a self-induction as possible, while the thick wire coil carries the whole current passing through the circuit.

The magnetic field due to the thick coil is therefore proportional to the current passing through the circuit, and that due to the thin coil is proportional to the P.D. between its terminals. The mutual force exerted by the two fields when the planes of the coils are at right angles is at any instant proportional to the product of the current passing through the circuit and the P.D. between its terminals. Thus the mutual action is at any instant proportional to the power which is being given to the circuit at that instant.

The movable coil is suspended so that its geometrical centre coincides with that of the fixed coil, and is attached to a torsion head by means of which the coils may be kept with their planes at right angles to each other, and the mutual force is measured by the angle through which the torsion head has to be rotated to do this.

The average mutual force would produce a steady deflection of the movable coil and is measured by the angle through which the torsion head is rotated to keep the two coils at right angles to each other.

If $\theta$ is the angle through which the torsion head is rotated, the power, $P$, is given by the equation—

$$P = k\theta$$

where $k$ is a constant which has to be determined experimentally.

The unit of power—the watt—is equal to $10^7$ ergs per second.

We shall deal further with this in Chap. XVII., which treats of various methods of measuring power.

## PROBLEMS ON CHAPTER V.

1. What energy is absorbed in five minutes by a coil of resistance 5 ohms and self-induction 0.05 henry if 100 volts is applied between its terminals, the frequency being 80 periods per second?

*Answer.* $228 \times 10^9$ ergs.

2. What is the power given to a series circuit whose self-induction is 0.05 henry, capacity 1 microfarad, and resistance 10 ohms, when a P.D. of 100 volts at a frequency of 50 periods per second is applied between its terminals?

*Answer.* 0.01 watt.

# CHAPTER VI.

## Composition of Periodic Curves of Different Amplitudes but of same Periodic Time.

### GRAPHICAL METHODS.

**30.** Consider a point $P$ moving in a counter-clockwise direction with uniform angular velocity $p$ round a circle of centre $O$ (Fig. 9). Suppose that at time $t = o$, $P$ is at the point $A$, and that at any subsequent time $t$ it occupies a position such that the angle $AOP = pt$. Let $AOA'$ and $BOB'$ be two diameters at right angles to each other, and let $PN$ be drawn at right angles to $AA'$. Let $OP = r$. Then—

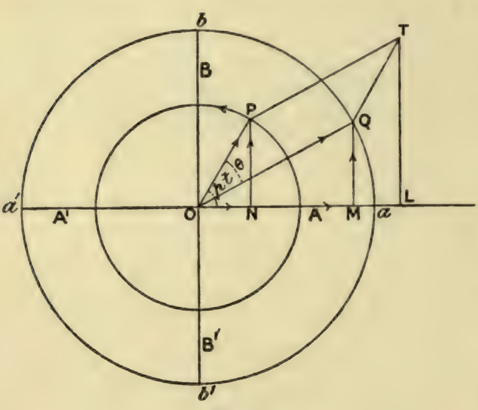


FIG. 9.

$$PN = OP \sin AOP$$
$$= r \sin pt$$

Therefore $PN$ is a sine function of the time and $r$ is its maximum value.

As $P$ travels round the circle the value of $PN$ changes from zero to its maximum value $OB$, diminishing to zero when $P$ reaches $A'$, after which it changes in sign, attaining a negative maximum when at $B'$, and completing its cycle at $A$.

If we plot a curve having values of $pt$ as abscissæ and the corresponding values of $PN$ as ordinates, we obtain the usual curve of sines shown in Fig. 3.

Suppose now that a second point $Q$ travels round a concentric circle of radius $r'$ starting from $a$ $t'$ seconds later than $P$ starts from $A$; then at time $t$ the position of $Q$ will be such that the

angle $aOQ = p\,(t - t')$, and if $QM$ is drawn at right angles to $Oa$, we have—

$$QM = OQ \sin p\,(t - t')$$
$$= r' \sin (pt - \theta)$$

where—

$$pt' = \theta = \text{angle } POQ$$

If we take the sum of $PN$ and $QM$, we get—

$$PN + QM = TL$$

where $T$ is determined by completing the parallelogram having $OP$ and $OQ$ as adjacent sides, and $L$ is the foot of the perpendicular from $T$ on $Oa$ produced.

Thus the sum of $PN$ and $QM$ is a sine function of the time having the same periodic time as either $PN$ or $QM$, but differing in phase from either $PN$ or $QM$, and having its maximum value given by the diagonal through $O$ of the parallelogram $POQT$.

Hence, if the maximum values of two alternating currents, or E.M.F.'s were represented by $OP$ and $OQ$ respectively, the maximum value of the current, or E.M.F. resulting from the two when superimposed, is determined by the law of composition of Vectors, and is the diagonal through $O$ of the parallelogram with $OP$ and $OQ$ as adjacent sides.

The corresponding instantaneous values are given by the ordinates $PN$, $QM$, and $TL$ respectively, as the parallelogram $POQT$ is rotated round $O$ with uniform angular velocity.

**31.** Suppose we consider the case of a single inductive circuit, whose self-induction is $L$ and resistance $r$. Let $E$ be the maximum value of the P.D. impressed between the terminals of the circuit and $I$ the maximum value of the current produced.

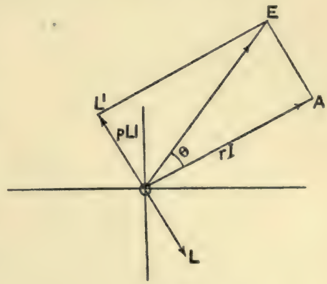


FIG. 10.

Let $OA$ (Fig. 10) represent in magnitude $rI$, that is the E.M.F. necessary to drive the maximum current against the resistance. This is in phase with the current. The maximum value of the E.M.F. due to self-induction is $pLI$ (see § 21, Chap. III.), and is represented by $OL$ drawn at right angles to $OA$ and 90° behind it. The maximum value of the E.M.F. necessary

to balance this is therefore given by $OL'$, where $OL' = OL$. The maximum value, $E$, of the P.D. is therefore given by $OE$, the diagonal of the parallelogram $L'OAE$.

Thus—

$$E^2 = r^2I^2 + p^2L^2I^2$$

or—

$$I = \frac{E}{\sqrt{r^2 + p^2L^2}}$$

Also the current lags behind the P.D. by an angle $\theta$ where—

$$\tan \theta = \frac{pL}{r}$$

Therefore, if the P.D. is given by $e = E \sin pt$, the current is given at any instant by—

$$i = \frac{E \sin (pt - \theta)}{\sqrt{r^2 + p^2L^2}}$$

which is the same as equation (11) Chap. III., which was obtained by other means.

The instantaneous values of $e$ and $ri$ are given by the lengths of the perpendiculars drawn from the points $E$ and $A$ to the base line as the rectangle $OAEL'$ revolves about $O$.

# CHAPTER VII.

Algebraic Representation of Vectors—Vector Addition—Products of Vectors.

## The Elements of Vector Algebra.

**32.** Any physical quantity which requires for its complete specification data regarding (1) its **magnitude,** (2) its **direction,** and (3) its **sense** along that direction is called a **vector** quantity. Quantities which are completely specified when their magnitudes only are given are called **scalar** quantities.

Mass and energy are examples of scalar quantities; velocity, acceleration, force, electric current, and electromotive force are examples of vector quantities.

**Vector.**—A vector quantity may be completely represented by a straight line drawn in a particular direction, the sense along the direction being shown by means of an arrowhead, and the line containing as many units of length as the quantity to be represented contains units of quantity.

We call a line drawn in this way a **vector,** *e.g.* the vector $OP$ (Fig. 11) may represent an electric current if its direction is represented by the direction of the line $OP$, its sense from $O$ to $P$ along this direction, and if $OP$ contains as many units of length as the number of amperes (or other unit of current) in the electric current.

**33. Equality of Vectors.**—Two vectors are equal if they contain the same number of units of length, are parallel to the same direction, and have the same sense; thus in Fig. 11 the vector $OP$ is equal to the vector $O'P'$, if the length of $OP$ equals that of $O'P'$, the two vectors being parallel and of the same sense. If the **sense** of a vector along a given direction is reversed, its sign is reversed, so that—

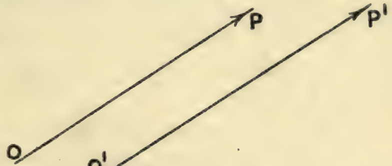


Fig. 11.

$$PO = -OP$$
$$\text{or } PO + OP = 0$$

**34. Composition of Vectors.**—The sum of two vectors $OP$ and $PQ$ (Fig. 12) is defined to be the vector $OQ$. The usual meaning of the word **sum** is here extended, and if we write—

$$OP + PQ = OQ$$

we should read $OP$ together with $PQ$ are equivalent to $OQ$. The addition of vectors is thus the same as the combination of forces, for if $OP$ and $PQ$ represent two forces, then $OQ$ represents their resultant; in fact, as we have already stated, a force is a vector quantity.

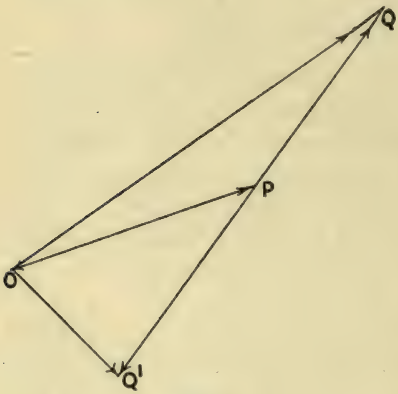

FIG. 12.

The difference of two vectors $OP$ and $PQ$ is defined to be the sum of the two vectors $OP$ and $QP$, and is, therefore, the vector $OQ'$ (Fig. 12) where $PQ' = PQ$, that is $PQ'$ equals $PQ$ in magnitude, but is drawn in the opposite sense. It may be well to emphasize here that the conditions of equality of two vectors in no way fixes the vectors in space: they must merely be of equal length parallel in direction and of the same sense; so long as these conditions are satisfied, the actual positions of the vectors in space is immaterial.

**35. Algebraic Expression for a Vector Quantity.**—Let $OP$ (Fig. 13) represent any vector quantity.

Through $O$ draw any two mutually perpendicular lines $xOx'$, $yOy'$. Draw $PN$ perpendicular to $xOx'$; the two vectors, $ON$ and $NP$, are then together equal to the vector $OP$. Any vector can thus be resolved into two component vectors parallel respectively to $xOx'$ and $yOy'$.

Now let us agree to represent unit vector along $Ox$ by $+1$; unit vector along $Ox'$ will then be represented by $-1$, since the sense is exactly opposite. Let us further represent unit vector along $Oy$ by $k$; then unit vector along $Oy'$ will be represented by $-k$.

If, then, $ON$ contains $a$ units of length, and $NP$ contains $b$ units, the vector $OP$ is given by—

$$\begin{aligned} OP &= ON + NP \\ &= a + kb \end{aligned}$$

and its magnitude is $\sqrt{a^2 + b^2}$; also its inclination $\theta$ to $Ox$ is given by—

$$\tan \theta = \frac{b}{a}$$

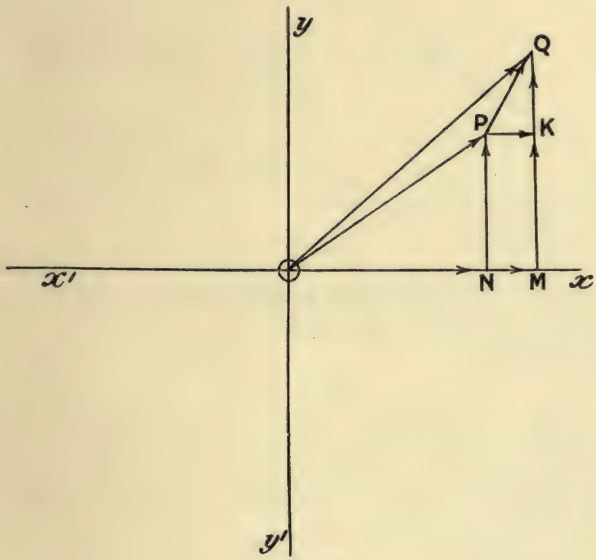


FIG. 13.

In the same way $-a + kb$ represents a vector lying in the quadrant $yOx'$; $-a - kb$, one lying in the quadrant $x'Oy'$; and $a - kb$, one lying in the quadrant $y'Ox$.

**36. Vector Addition.**—If $PQ$ (Fig. 13) is another vector whose components parallel to $Ox$ and $Oy$ are respectively $PK = NM = a'$ and $KQ = b'k$, then the vector $PQ$ is represented by $a' + kb'$, and the vector $OQ$, which is the sum of the vectors $OP$ and $PQ$, is given by—

$$\begin{aligned} OQ &= OM + MQ \\ &= (ON + NM) + (MK + KQ) \\ &= (a + a') + (kb + kb') \\ &= (a + a') + k(b + b') \end{aligned}$$

This gives the law of vector addition, and inherently contains that of subtraction also, the difference of the vectors $OP$ and $PQ$ being represented by—

$$(a - a') + k(b - b')$$

## MULTIPLICATION OF VECTORS.

**37. Vector Products.**—Consider any two vectors $OP$ and $OQ$ (Fig. 14), and let $OP = ON + NP$, where $NP$ is at right angles to $OQ$.

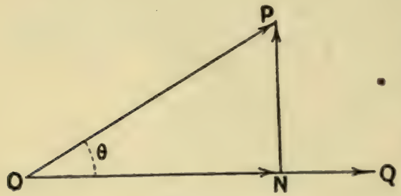

FIG. 14.

There are two products to take into consideration, viz. the product—

$$OQ \,.\, ON$$

and the product—

$$OQ \,.\, NP$$

To interpret these, suppose that the vector $OP$ represents a displacement produced by the action of a force which is represented by the vector $OQ$.

The product—

$$OQ \,.\, ON$$

then represents the **work** done by the force in moving its point of application from the point $O$ to the point $N$. This product is essentially **scalar,** since it is the increase of energy of the system on which the force $Q$ acts.

The product—

$$OQ \,.\, NP$$

represents the **moment** of the force about an axis through the point $P$ at right angles to both $OP$ and $NP$, and is **a vector at right angles to the plane** $OPN$.

It is the scalar product of two vectors with which we are chiefly interested, so we leave the vector product for the present.

By reference to Fig. 14, we see that

$$OQ \,.\, ON = OQ \,.\, OP \cos \theta$$

where $\theta$ is the angle between the directions of $OQ$ and $OP$ respectively; hence **the scalar product of two vectors is the product of their lengths multiplied by the cosine of the angle between their respective directions.**

Let the vector $OP$ be represented algebraically by—

$$a + kb$$

and the vector $OQ$ by—

$$a' + kb'$$

then (see Fig. 13) the magnitude of $OP$ is—

$$\sqrt{a^2 + b^2}$$

and that of $OQ$ is—

$$\sqrt{a'^2 + b'^2}$$

and the angle $\theta$ between their directions is—

$$\theta = \theta_1 - \theta_2$$

where—

$$\tan \theta_1 = \frac{b}{a}$$

and—

$$\tan \theta_2 = \frac{b'}{a'}$$

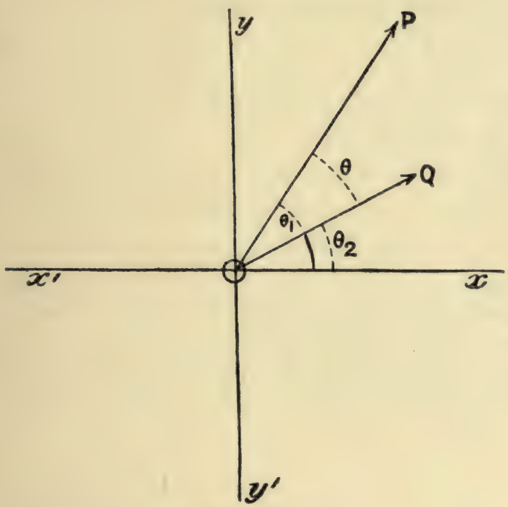


FIG. 15.

therefore (see Fig. 15)—

$$\begin{aligned}\cos \theta &= \cos (\theta_1 - \theta_2)\\ &= \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2\\ &= \frac{a}{\sqrt{a^2 + b^2}} \cdot \frac{a'}{\sqrt{a'^2 + b'^2}} + \frac{b}{\sqrt{a^2 + b^2}} \cdot \frac{b'}{\sqrt{a'^2 + b'^2}}\end{aligned}$$

that is—

$$\begin{aligned}OQ \,.\, ON &= OQ \,.\, OP \cos \theta\\ &= \sqrt{a^2 + b^2}\sqrt{a'^2 + b'^2} \cos \theta\\ &= aa' + bb'\end{aligned}$$

Thus **the scalar product of the two vectors**

$$a + kb$$

and—

$$a' + kb'$$

is—

$$aa' + bb'$$

**38. On *k* as an Operator.**—A vector whose length is $a$ along or parallel to $Ox$ is represented by $a$; whereas a vector of the same length, whose direction is parallel to $Oy$, is represented by $ka$. We may thus regard $k$ as an operator which, when it operates upon a vector along $Ox$, has the effect of turning the vector through a right angle in the positive (counter-clockwise) direction of rotation without altering its length. The effect of similarly operating on $ka$ must, therefore, to be consistent, be to turn the vector $ka$ through a right angle in the positive direction without altering its length; that is, it becomes a vector of length $a$ along $Ox'$; that is, it becomes $-a$.

We thus have—

$$k \cdot ka = k^2 a = -a$$

or—

$$k^2 = -1$$

that is, when using the symbol $k$ in algebraical processes, we must regard it as having the properties of the imaginary $\sqrt{-1}$.

If we similarly operate with $k$ on the vector $-a$, we get $-ka$, a vector of length $a$ along $Oy'_1$; and operating again, we get $-k^2a$, or $+a$, a vector of length $a$ along $Ox$.

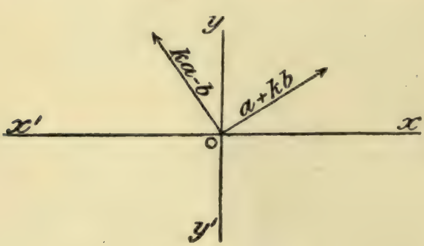


FIG. 16.

Again, operating on any vector

$$a + kb$$

we get—

$$\begin{aligned} k(a + kb) &= ka + k^2b \\ &= ka - b \end{aligned}$$

which (see Fig. 16) is the vector $a + kb$ turned through a right angle in the positive direction of rotation, its length being unaltered.

**That is, if *OP* is any vector, then *k . OP* is the**

**vector resulting from a rotation of the vector *OP* through a right angle in the positive direction of rotation without any alteration of its length.**

By similar reasoning we can show that $-k \,.\, OP$ represents the vector resulting from a rotation of the vector $OP$ through a right angle in the negative (clockwise) direction of rotation.

**39. Definition of Lag and Lead.**—A vector $OP$ is said to lead before, or lag behind, a vector $OQ$, according as the least amount of rotation necessary to bring $OP$ into the same direction as $OQ$ is in the negative or positive direction.

It follows, therefore, that the vector $k \,.\, OP$ leads before the vector $OP$ by a right angle, and that the vector $-k \,.\, OP$ lags behind the vector $OP$ by a right angle.

## PROBLEMS ON CHAPTER VII.

1. Draw diagrams showing the following vectors :—

$$k, \quad 1+k, \quad 2+3k, \quad -2-3k, \quad -2+3k.$$

2. What is the sum of all the vectors in Question 1 ?

*Answer.* $-1+5k$.

3. What is the product of each of the following pairs of vectors ?

$$\left.\begin{matrix}2+3k\\3+3k\end{matrix}\right\} \qquad \left.\begin{matrix}2+3k\\3-2k\end{matrix}\right\} \qquad \left.\begin{matrix}-2-3k\\3-2k\end{matrix}\right\} \qquad \left.\begin{matrix}-2-3k\\-3-2k\end{matrix}\right\}$$

*Answers.* 15; 0; 0; 12.

# CHAPTER VIII.

The Calculation of Electric Currents in Reactive Circuits—Series Circuits—Parallel Circuits—Mutually Reactive Circuits.

## The Application of Vector Algebra to Alternating-current Problems.

**40.** Before attempting to read this and subsequent chapters, the student is urged to make himself thoroughly conversant with the contents of the foregoing chapter on Vector Algebra. All that is necessary for a comprehensive study of its subsequent applications is there given, and the method is so much simpler and more instructive than the more ponderous methods involving the calculus and differential equations, that no apology is necessary for its introduction. After thoroughly mastering the method, which will require a concentrated effort for but a short time, subsequent reading will be simplicity itself, and the student should experience little or no difficulty in solving most alternating-current problems by its help. One feature, perhaps, more than any other stamps the method of vector algebra as being *par excellence* the method for the solution of alternating-current problems; that is, that while it obviates a *profound mathematical knowledge*, it does not in any way save *thought*. Each problem must be thoroughly viewed and understood in its physical aspects before the method of vector algebra can be applied. In the following applications we shall endeavour to emphasize the value of the method by a judicious selection of problems, and by deducing the vector equations from physical considerations only.

**41.** Before proceeding to apply the method to specific problems, we must recall three important propositions.

**Proposition 1.**—The maximum value of the induced E.M.F. due to self-induction in a circuit is $pLI$, where $I$ is the maximum value of the current in the circuit, $L$ is the self-induction of the circuit, and $p = 2\pi n$, where $n$ is the frequency of

the current. Also, the E.M.F. of self-induction **lags a right angle behind the current.** For the proof of this, see Chap. III., § 21.

**Proposition 2.**—If a condenser of capacity $C$ is placed in an alternating-current circuit, an E.M.F. whose maximum value is $\frac{I}{pC}$, where $I$ is the maximum value of the current flowing through the circuit, is generated in consequence, and this capacity E.M.F. **leads before the current by a right angle.** The proof of this is given in Chap. IV., § 22.

**Proposition 3.**—If two circuits, $A$ and $B$, have a mutual induction, $M$, the maxima values of the consequent E.M.F.s in the circuits $A$ and $B$ are respectively $pMI_2$ and $pMI_1$, where $I_1$ is the maximum value of the current in the circuit $A$, and $I_2$ that in the circuit $B$; and **these induced E.M.F.s lag a right angle behind the currents $I_2$ and $I_1$ respectively.**

**Proof.**—Let the current in the circuit $B$ be given by—

$$i_2 = I_2 \sin pt$$

then the E.M.F. $e_1$ induced in the circuit $A$ due to mutual induction, is given by—

$$\begin{aligned} e_1 &= -M\frac{d}{dt}(I_2 \sin pt) \\ &= -pMI_2 \cos pt \\ &= pMI_2 \sin\left(pt - \frac{\pi}{2}\right) \end{aligned}$$

The maximum value of this is $pMI_2$, and it lags by an angle $\frac{\pi}{2}$ behind the current $i_2$; similarly the E.M.F. induced by mutual induction in circuit $B$ lags by an angle $\frac{\pi}{2}$ behind the current $i_1$. Thus the proposition is established.

Having established these propositions, we are in a position to proceed to the solution of some typical problems by the aid of the vector calculus.

## Circuits containing Resistance and Self-induction only.

**42. To determine the Current flowing in a Circuit of Resistance $r$ and Self-induction $L$,**

**when an Alternating P.D. whose Maximum Value is $e$ and Frequency is $n$, is applied between its Terminals.**

Let $p = 2\pi n$,

and $I$ be a vector representing the maximum value of the current flowing through the circuit.

Then by proposition 1 and § 38 the vector representing the maximum value of the E.M.F. of self-induction in the circuit in which the current $I$ is flowing is—

$$-kpLI$$

The applied potential difference has to drive the current against the resistance of the circuit and also to balance the E.M.F. of self-induction. The vector $e$ has therefore to be capable of resolution into two components—one equal to $rI$, and the other equal to $+kpLI$. We thus arrive at the vector equation—

$$rI + kpLI = e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is essentially a vector equation of E.M.F.s.

If we reduce the lengths of each of these vectors in the ratio $\sqrt{2} : 1$, we may regard the vectors as representing root mean square values; but this would not produce any change in equation (1), which only depends upon ratios of magnitudes of vectors, and therefore holds good if each vector in the equation is multiplied by the same constant.

Let $OP$ (Fig. 17) represent the vector $rI$.

Then the vector $OQ' = -kpLI$ represents the E.M.F. of self-induction; and the vector $PQ = +kpLI$, which is equal in length to $OQ'$, parallel to it, but of opposite sense, is the vector representing the E.M.F. necessary to overcome self-induction; therefore the vector

$$OQ = OP + PQ$$

is the vector representing the applied potential difference $e$.

Further, the vector $OP$, which is in phase with the current, lags behind the vector representing $e$ by an angle, $\theta$, where—

$$\tan \theta = \frac{pL}{r}$$

Also (see § 35) the **magnitude** of $e$ is given by—

$$e = \sqrt{r^2I^2 + p^2L^2I^2}$$
$$= I\sqrt{r^2 + p^2L^2}$$

that is—

$$I = \frac{e}{\sqrt{r^2 + p^2L^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and if the instantaneous value of the P.D. is given by $e \sin pt$, the instantaneous value of the current is given by—

$$i = \frac{e \sin (pt - \theta)}{\sqrt{r^2 + p^2L^2}}$$

Equation (2), it must be noticed, is not the same as equation (1); it is one of several deductions from equation (1), and

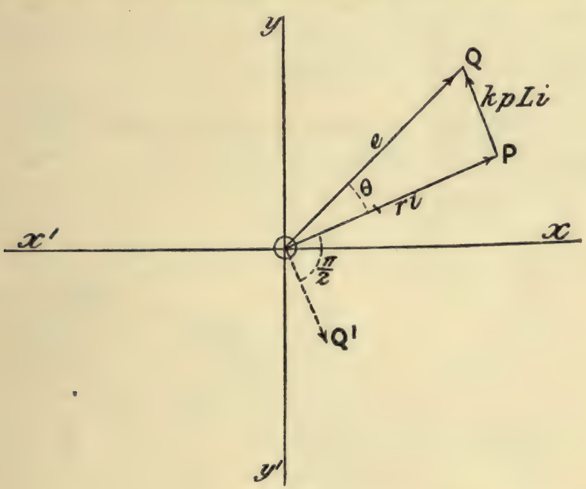


FIG. 17.

while (1) is a vector equation involving directions as well as magnitudes, (2) involves magnitudes only.

Equation (2) gives us the maximum or the R.M.S. value of the current when the maximum or the R.M.S. value of the P.D. is given. It also tells us that the impedance (see § 21) of the circuit is—

$$\sqrt{r^2 + p^2L^2}$$

Referring again to equation (1), it may be written—

$$I(r + kpL) = e$$

or

$$\begin{aligned} I &= \frac{e}{r + kpL} \\ &= \frac{(r - kpL)e}{(r + kpL)(r - kpL)} \\ &= \frac{(r - kpL)e}{r^2 + p^2L^2} \end{aligned}$$

since $k^2 = -1$

$$= \frac{r}{r^2 + p^2L^2} \cdot e - \frac{pL}{r^2 + p^2L^2} \cdot ke \quad . \quad . \quad (3)$$

which is a current-vector equation, and states that the current $I$ can be resolved into a component

$$\frac{r}{r^2 + p^2L^2} \cdot e$$

parallel to the direction of $e$, that is, in phase with $e$, and a component

$$\frac{pI}{r^2 + p^2L^2} \cdot (-ke)$$

at right angles to the direction of $e$, and lagging behind it. This is shown graphically in Fig. 18, where $OP$ is the current vector,

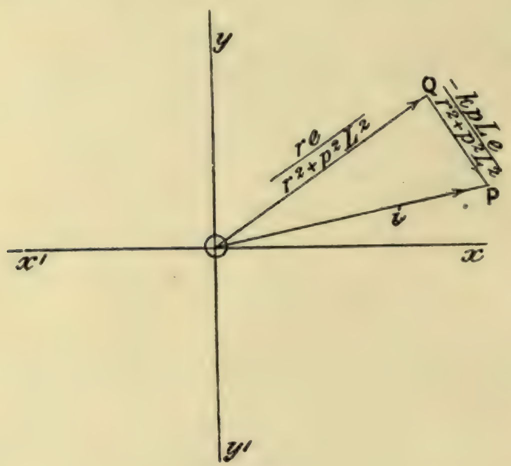


Fig. 18.

and $OQ$ and $QP$ its components respectively along and at right angles to the direction of $e$. It should be noticed that the product

$$(r + kpL)(r - kpL)$$

which equals

$$r^2 + p^2L^2$$

since $k = -1$, is not a vector product; the quantities $r$, $p$, and $L$ are all scalar, and must not be treated as vectors.

We shall frequently make use of the vector equations (1) and (3), so they should be thoroughly understood; in fact, to proceed further without first mastering them would be useless.

### 43. To find the Equivalent Resistance, Self-induction, and Impedance of *n* Inductive Circuits connected in Series.

Let $r_1, r_2, r_3, \ldots r_n$ be the ohmic resistances of the several circuits.

Let $L_1, L_2, L_3, \ldots L_n$ be their inductances.

$e_1, e_2, e_3, \ldots e_n$ be the vectors representing the R.M.S. potential differences between their respective terminals.

Let $i$ be the vector representing the R.M.S. current flowing through the series circuit.

Let $p = 2\pi n$, where $n$ is the frequency of the current.

Then, the vector $e$ representing the potential difference between the extreme terminals of the series circuit is given by the vector equation—

$$e = e_1 + e_2 + e_3 + \ldots + e_n$$

But by equation (1)—

$$\begin{aligned} e_1 &= r_1 i + kpL_1 i \\ e_2 &= r_2 i + kpL_2 i \\ e_3 &= r_3 i + kpL_3 i \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ e_n &= r_n i + kpL_n i \end{aligned}$$

Now, all the $ri$'s are in the same direction, viz. along the current vector, as also are all the $kpLi$'s, viz. leading a right angle in front of the current vector; they can, therefore, be respectively added together numerically, and we get—

$$\begin{aligned} e &= e_1 + e_2 + e_3 + \ldots + e_n \\ &= (r_1 + r_2 + r_3 + \ldots + r_n)i \\ &\quad + kp(L_1 + L_2 + L_3 + \ldots + L_n)i \quad . \quad . \quad (4) \end{aligned}$$

But if $R$ and $L$ are the equivalent resistances and self-induction of the complete circuit, we have by equation (1)—

$$e = Ri + kpLi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Comparing equations (4) and (5), we see that

$$\begin{aligned} R &= r_1 + r_2 + r_3 + \ldots + r_n \\ L &= L_1 + L_2 + L_3 + \ldots + L_n \end{aligned}$$

and the impedance $I$ is given by—

$$I = \sqrt{R^2 + p^2L^2}$$

**44. To find the Current passing through a Circuit containing a Resistance of *r* Ohms in Series with a Capacity of *C* Farads, when an Alternating P.D. of *e* R.M.S. Volts having Frequency *n* is applied between its Terminals.**

Let $p = 2\pi n$

and $i$ = the R.M.S. current in amperes.

The E.M.F. necessary to drive the current against the resistance of the circuit is—

$$ri$$

The E.M.F. due to capacity is—

$$\frac{ki}{pC}$$

by §§ 38 and 40, since it leads a right angle before the current; therefore the E.M.F. required to balance it is—

$$-\frac{ki}{pC}$$

The applied P.D. has therefore to be capable of resolution into the two components, $ri$ and $-\frac{ki}{pC}$; thus we arrive at the vector equation—

$$ri - \frac{ki}{pC} = e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This equation is represented graphically in Fig. 19, in which $OP$ is the vector $ri$, $OQ'$ the vector representing the E.M.F. due to capacity, $PQ$ the vector representing the E.M.F. necessary to balance that due to capacity, and $OQ$ that representing the applied potential difference.

The magnitude of $e$ is, by § 35, given by—

$$e = \sqrt{r^2 + \frac{1}{p^2C^2}} \cdot i$$

therefore—

$$i = \frac{e}{\sqrt{r^2 + \frac{1}{p^2C^2}}}$$

Equation (6) shows that the current leads before the P.D. by an angle, $\theta$, given by—

$$\tan\theta = \frac{1}{pCr}$$

and if the P.D. is given at any instant by—

$$E \sin pt$$

the corresponding current is given by—

$$\frac{E \sin (pt + \theta)}{\sqrt{r^2 + \frac{1}{p^2C^2}}}$$

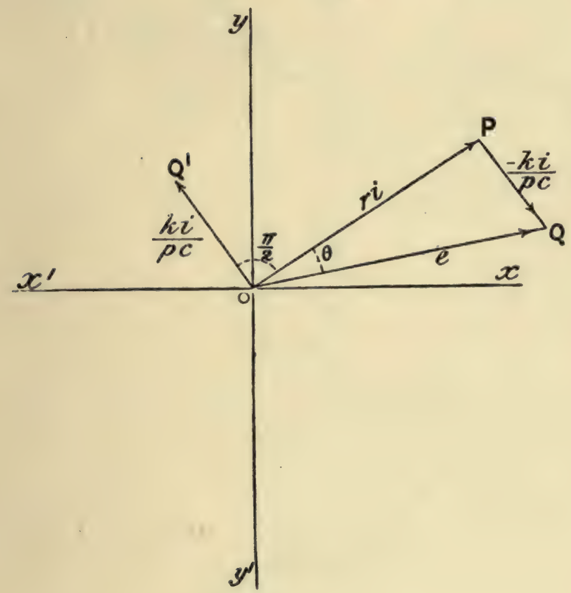


Fig. 19.

Equation (6) may be changed into a current-vector equation, thus—

$$i = \frac{e}{r - \frac{k}{pC}}$$

$$= \frac{\left(r + \frac{k}{pC}\right)e}{r^2 + \frac{1}{p^2C^2}}$$

$$= \frac{r}{r^2 + \frac{1}{p^2C^2}} \cdot e + \frac{\frac{1}{pC}}{r^2 + \frac{1}{p^2C^2}} \cdot ke \quad . \quad . \quad . \quad . \quad (7)$$

This is the current-vector equation, and is represented in

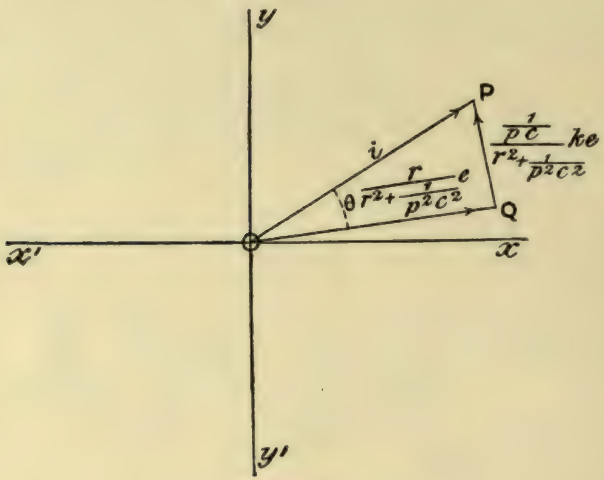


FIG. 20.

Fig. 20, which shows the components of the current along and at right angles to the P.D. $e$.

## COMBINATION OF RESISTANCE, SELF-INDUCTION, AND CAPACITY IN SERIES.

**45.** Consider a circuit consisting of a coil of wire, having a resistance $r$, and self-induction $L$, placed in series with a condenser of capacity $C$. Let an alternating P.D. $e$, of frequency $n$, be applied between the extreme terminals of the circuit, and cause an alternating current $i$ to flow through it, and let $p = 2\pi n$.

The P.D. must have a component $ri$ in the direction of the current, to drive the current against the ohmic resistance $r$, a component $kpLi$, to balance the E.M.F. $-kpLi$ of self-induction, and a component $-\frac{ki}{pC}$, to balance the E.M.F. $+\frac{ki}{pC}$ due to capacity.

We therefore have the vector equation of E.M.F.'s

$$ri + kpLi - \frac{ki}{pC} = e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

or

$$ri + k\left(pL - \frac{1}{pC}\right)i = e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This equation is graphically represented in Fig. 21, where $OP$ is the vector $ri$, $OL$ the vector $-kpLi$, $OC$ the vector $\frac{ki}{pC}$, $PQ$ the vector $-(OL + OC)$, and $OQ$ the vector $e$.

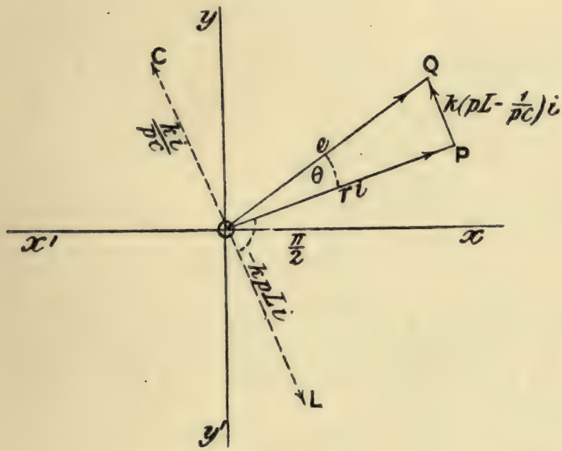


Fig. 21.

If $OL$ is greater than $OC$, the current lags behind the P.D. by an angle $\theta$, where

$$\tan\theta = \frac{pL - \frac{1}{pC}}{r}$$

If $OC$ is greater than $OL$, the current leads before $e$ by an angle given by—

$$\tan\theta = \frac{\frac{1}{pC} - pL}{r}$$

If $OL$ equals $OC$, that is, if—

$$pL = \frac{1}{pC}$$

the current and P.D. are in phase.

**Reactance.**—We shall denote the quantity $pL - \frac{1}{pC}$ by the letter $s$, and call it the reactance of the circuit.

Equation (9) then becomes—

$$r\dot{i} + ks\dot{i} = \dot{e} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If the P.D. is given at any instant by—

$$E \sin pt$$

the corresponding current is given by—

$$\frac{E \sin (pt - \theta)}{\sqrt{r^2 + s^2}}$$

From equation (10) we get—

$$\begin{aligned} \dot{i} &= \frac{\dot{e}}{r + ks} \\ &= \frac{\dot{e}(r - ks)}{r^2 + s^2} \\ &= \frac{r}{r^2 + s^2} \cdot \dot{e} - \frac{s}{r^2 + s^2} \cdot k\dot{e} \quad . \quad . \quad . \quad . \quad (11) \end{aligned}$$

Equations (10) and (11) are equations (1) and (3) extended to the case in which the circuit contains both self-induction and capacity, the former being an equation of E.M.F.s, and the latter an equation of currents.

The magnitude of $\dot{e}$ is given by—

$$e = \sqrt{r^2 + s^2} \cdot i$$

whence—

$$i = \frac{e}{\sqrt{r^2 + s^2}}$$

and the impedance of the circuit is—

$$\sqrt{r^2 + s^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

## SERIES CIRCUITS.

**46.** Suppose that $m$ reactive circuits are connected in series, and that an alternating P.D. $\dot{e}$ is applied between the extreme terminals of the combination. We propose to determine the equivalent resistance $R$, and the equivalent reactance $S$, of the combination.

Let the resistances of the circuits be respectively $r_1, r_2, r_3, \ldots r_m$, and their respective reactances $s_1, s_2, s_3, \ldots s_m$; and let the potential differences between the terminals of the respective circuits be $e_1, e_2, e_3, \ldots e_m$, and $i$ the current common to all the circuits.

We then have, by applying the equation (10) to each circuit in succession—

$$\begin{aligned} r_1i + ks_1i &= e_1 \\ r_2i + ks_2i &= e_2 \\ r_3i + ks_3i &= e_3 \\ &\cdots \\ r_mi + ks_mi &= e_m \end{aligned}$$

Therefore—

$$\begin{aligned} e &= e_1 + e_2 + e_3 + \ldots + e_m \\ &= (r_1 + r_2 + r_3 + \ldots + r_m)i \\ &\qquad + k(s_1 + s_2 + s_3 + \ldots + s_m)i \quad \ldots \quad (13) \end{aligned}$$

But, applying equation (10) to the combination, we have—

$$e = Ri + kSi \quad \ldots \ldots \quad (14)$$

Thus, by comparison of (13) and (14), we see that

and
$$\left.\begin{aligned} R &= r_1 + r_2 + r_3 + \ldots + r_m \\ S &= s_1 + s_2 + s_3 + \ldots + s_m \end{aligned}\right\} \quad \ldots \quad (15)$$

that is, the resistance and reactance of the series combination are, respectively, the sums (algebraic) of the resistances and reactances of the constituent circuits.

## Parallel Circuits.

**47.** Suppose that $m$ reactive circuits are connected in parallel, and that it is required to determine the equivalent resistance, reactance, and impedance of the combination. It is better to subdivide this problem into two distinct cases, according as mutual induction between the circuits is not, or is, taken into consideration.

Case 1.—**Mutual Induction neglected.**

Let the resistances and reactances of the individual circuits be $r_1, r_2, r_3, \ldots r_m$ respectively, and $s_1, s_2, s_3, \ldots s_m$ respectively;

and let the P.D. common to each circuit be $e$, the currents in the respective circuits being $i_1$, $i_2$, $i_3$, . . . $i_m$.

Then, applying equation (11) to each circuit in succession, we have the vector-current equations—

$$i_1 = \frac{r_1}{r_1^2 + s_1^2} \cdot e - \frac{s_1}{r_1^2 + s_1^2} \cdot ke$$

$$i_2 = \frac{r_2}{r_2^2 + s_2^2} \cdot e - \frac{s_2}{r_2^2 + s_2^2} \cdot ke$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$i_m = \frac{r_m}{r_m^2 + s_m^2} \cdot e - \frac{s_m}{r_m^2 + s_m^2} \cdot ke$$

But the current $i$ in the main circuit is the vector sum of the currents in the several branches of the parallel circuit; therefore we have the vector equation—

$$\begin{aligned} i &= i_1 + i_2 + i_3 + \ldots + i_m \\ &= \left(\frac{r_1}{r_1^2 + s_1^2} + \frac{r_2}{r_2^2 + s_2^2} + \ldots + \frac{r_m}{r_m^2 + s_m^2}\right)e \\ &\quad - \left(\frac{s_1}{r_1^2 + s_1^2} + \frac{s_2}{r_2^2 + s_2^2} + \ldots + \frac{s_m}{r_m^2 + s_m^2}\right)ke \quad . \ (16) \end{aligned}$$

Or, writing—

$$\left.\begin{aligned} A &= \frac{r_1}{r_1^2 + s_1^2} + \frac{r_2}{r_2^2 + s_2^2} + \ldots + \frac{r_m}{r_m^2 + s_m^2} \\ B &= \frac{s_1}{r_1^2 + s_1^2} + \frac{s_2}{r_2^2 + s_2^2} + \ldots + \frac{s_m}{r_m^2 + s_m^2} \end{aligned}\right\} \quad . \ . \ (17)$$

we get—

$$\begin{aligned} i &= Ae - kBe \\ &= \frac{(A^2 + B^2)e}{A + kB} \\ &= \frac{e}{\dfrac{A}{A^2 + B^2} + k \cdot \dfrac{B}{A^2 + B^2}} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (18) \end{aligned}$$

And by comparison of this with the equation for the combination—

$$i = \frac{e}{R + kS} \quad . \ . \ . \ . \ . \ . \ . \ . \ (19)$$

we see that the equivalent resistance $R$ and equivalent reactance $S$ are respectively given by—

and

$$\left.\begin{aligned} R &= \frac{A}{A^2 + B^2} \\ S &= \frac{B}{A^2 + B^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Also, the equivalent impedance $I$ of the multiple circuit is given by—

$$\begin{aligned} I &= \sqrt{R^2 + S^2} \\ &= \frac{1}{\sqrt{A^2 + B^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21) \end{aligned}$$

where $A$ and $B$ are defined by (17).

A particular case worthy of attention is that in which the arrangement consists of two circuits only, one of which contains a resistance $r_1$ and a self-induction $L$, while the other contains a resistance $r_2$ and a capacity $C$.

In this case—

$$\begin{aligned} s_1 &= pL \\ s_2 &= -\frac{1}{pC} \end{aligned}$$

so that—

$$A = \frac{r_1}{r_1^2 + p^2L^2} + \frac{r_2}{r_2^2 + \dfrac{1}{p^2C^2}}$$

$$B = \frac{pL}{r_1^2 + p^2L^2} - \frac{\dfrac{1}{pC}}{r_2^2 + \dfrac{1}{p^2C^2}}$$

Also, if $i_1$ and $i_2$ are the currents in the two branches respectively—

$$i_1 = \frac{e}{\sqrt{r_1^2 + p^2L^2}}$$

$$i_2 = \frac{e}{\sqrt{r_2^2 + \dfrac{1}{p^2C^2}}}$$

and the main current $i$ is given by—

$$i = (A - kB)e$$

$$= \left\{\left(\frac{r_1}{r_1^2 + p^2L^2} + \frac{r_2}{r_2^2 + \frac{1}{p^2C^2}}\right) - k\left(\frac{pL}{r_1^2 + p^2L^2} - \frac{\frac{1}{pC}}{r_2^2 + \frac{1}{p^2C^2}}\right)\right\}e$$

If $r_2 = o$, and $p$ is large, $i_1$ is small, and $i_2$ large, so that the greater portion of the current passes through the condenser circuit.

If, in addition, $r_1 = o$, the main current is given by—

$$i = -\left(\frac{1}{pL} - pC\right)e$$

If, further, $\frac{1}{pL} = pC$, the main current is zero, while the currents in the two branches are given by—

$$i_1 = i_2 = \frac{e}{pL} = pCe$$

that is, a current will circulate round the branch circuits while there is no current in the main circuit. It is to be noted that $r_1$ and $r_2$ can never be zero, although they may be very small, in which case the current in the main circuit will be small compared with those in the branch circuits.

CASE 2. **Mutual Induction taken into Consideration.**—This case is somewhat complicated, since, if we consider any particular branch of the parallel circuit, the E.M.F. which drives the current against its ohmic resistance is the resultant of $m + 1$ distinct E.M.F.s, viz. the applied P.D., the E.M.F. due to the self-induction of that circuit, and the E.M.F.s due to the mutual inductions between it and the remaining $m - 1$ branch circuits.

Let the mutual inductions of the several pairs of circuits be $M_{1,2}$, $M_{1,3}$, . . . $M_{pq}$, . . . where the suffixes denote the two circuits to which $M$ refers. Since the mutual induction between two circuits is reciprocal relation, $M_{pq} = M_{qp}$, where $p$ and $q$ are any different integers from $o$ to $m$. Let the remaining notation be as in Case 1. Then considering the circuit 1, the applied P.D. must furnish $m + 1$ components, one equal to $r_1i_1$, to drive the current against the ohmic resistance of the circuit; a second given by $ks_1i_1$, to balance the reactive E.M.F. of the circuit; a third given by $kpM_{1,2}i_2$, to balance the E.M.F. due to mutual induction of the

circuits 1 and 2 (see Proposition 3); a fourth given by $kpM_{1,3}i_3$, to balance the E.M.F. due to the mutual induction of the circuits 1 and 3, and so on.

Thus the vector equation of E.M.F.s in circuit 1 is—

$$r_1i_1 + ks_1i_1 + kpM_{1,2}i_2 + kpM_{1,3}i_3 + \ldots + kpM_{1,m}i_m = e$$

Similarly for circuit 2—

$$kpM_{2,1}i_1 + r_2i_2 + ks_2i_2 + kpM_{2,3}i_3 + \ldots + kpM_{2,m}i_m = e$$

Similarly for circuit 3—

$$kpM_{3,1}i_1 + kpM_{3,2}i_2 + r_3i_3 + ks_3i_3 + \ldots + kpM_{3,m}i_m = e \quad . \quad (22)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Similarly for circuit $m$—

$$kpM_{m1}i_1 + kpM_{m2}i_2 + kpM_{m3}i_3 + \ldots + r_mi_m + ks_mi_m = e$$

These are $m$ simultaneous equations from which to determine the currents $i_1$, $i_2$, $i_3$, . . . $i_m$, whence, by substitution, the current $i$ in the main circuit can be obtained by means of the vector equation—

$$i = i_1 + i_2 + i_3 + \ldots + i_m$$

In this way we shall arrive, in any particular case, at an equation of the form—

$$i = Pe + kQe$$

where $P$ and $Q$ are independent of $k$ and $e$. Having obtained this equation, the equivalent resistance and reactance can be found in the usual manner.

There is no mathematical difficulty in the formation of the vector equations (22), only great care should be taken in expressing exactly the physical relations existing between the several circuits. There is, however, a more extensive knowledge of mathematics required for the general solution of the equations there given, so we pass it over for the present, and content ourselves with a consideration of the simple example of two mutually inductive circuits connected in parallel.

The reader with sufficient mathematical knowledge may refer to the Appendix for the general solution.

## Two Reactive and Mutually Inductive Circuits connected in Parallel.

**48.** In this case the vector equations to be solved are—

$$\left.\begin{aligned} (r_1 + ks_1)i_1 + kpM_{1,2}i_2 &= e \\ kpM_{2,1}i_1 + (r_2 + ks_2)i_2 &= e \\ \text{and} \qquad i_1 + i_2 &= i \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

whence, on putting—

$$M_{1,2} = M_{2,1} = M$$

and solving as simultaneous equations for $i_1$ and $i_2$, we get—

$$\left.\begin{aligned} \{r_1r_2 - s_1s_2 + p^2M^2 + k(r_1s_2 + r_2s_1)\}i_1 &= \{r_2 + k(s_2 - pM)\}e \\ \{r_1r_2 - s_1s_2 + p^2M^2 + k(r_1s_2 + r_2s_1)\}i_2 &= \{r_1 + k(s_1 - pM)\}e \end{aligned}\right\} . \ (24)$$

Putting, for shortness—

$$\left.\begin{aligned} r_1r_2 - s_1s_2 + p^2M^2 &= P \\ r_1s_2 + r_2s_1 &= Q \end{aligned}\right\}$$

these equations become—

$$\left.\begin{aligned} (P + kQ)i_1 &= \{r_2 + k(s_2 - pM)\}e \\ (P + kQ)i_2 &= \{r_1 + k(s_1 - pM)\}e \end{aligned}\right\} \quad . \quad . \quad . \quad (25)$$

which, by multiplying both sides of the equations by $P - kQ$, and simplifying, may be written—

$$\left.\begin{aligned} (P^2 + Q^2)i_1 &= [Pr_2 + Q(s_2 - pM) + k\{P(s_2 - pM) - r_2Q\}]e \\ (P^2 + Q^2)i_2 &= [Pr_1 + Q(s_1 - pM) + k\{P(s_1 - pM) - r_1Q\}]e \end{aligned}\right\} . \ (26)$$

These are the vector-current equations giving the components of $i_1$ and $i_2$ along and at right angles to $e$. By addition, we have—

$$\begin{aligned} (P^2 + Q^2)i = [&P(r_1 + r_2) + Q(s_1 + s_2 - 2pM) \\ &+ k\{P(s_1 + s_2 - 2pM) - Q(r_1 + r_2)\}]e \ . \ (27) \end{aligned}$$

which gives the components of the main current along and at right angles to $e$; thus the component in phase with $e$ is—

$$\frac{P(r_1 + r_2) + Q(s_1 + s_2 - 2pM)}{P^2 + Q^2} \cdot e$$

and the component at right angles to $e$, or the wattless component, as it is called, is—

$$\frac{P(s_1 + s_2 - 2pM) - Q(r_1 + r_2)}{P^2 + Q^2} \cdot e$$

Adding equations (25), we get—

$$(P + kQ)i = \{r_1 + r_2 + k(s_1 + s_2 - 2pM)\}e$$

and multiplying this throughout by—

$$r_1 + r_2 - k(s_1 + s_2 - 2pM)$$

it becomes—

$$\begin{aligned}[P(r_1+r_2)+Q(s_1+s_2-2pM)-k\{P(s_1+s_2-2pM)-Q(r_1+r_2)\}]i \\ = \{(r_1+r_2)^2+(s_1+s_2-2pM)^2\}e\end{aligned}$$

which shows that the equivalent resistance $R$, and reactance $S$, of the parallel circuit are given by—

$$\left.\begin{aligned} R &= \frac{P(r_1 + r_2) + Q(s_1 + s_2 - 2pM)}{(r_1 + r_2)^2 + (s_1 + s_2 - 2pM)^2} \\ S &= \frac{Q(r_1 + r_2) - P(s_1 + s_2 - 2pM)}{(r_1 + r_2)^2 + (s_1 + s_2 - 2pM)^2} \end{aligned}\right\} \quad . \quad . \quad . \quad (28)$$

and the equivalent impedance $I$ is given by—

$$\begin{aligned} I^2 &= R^2 + S^2 \\ &= \frac{P^2 + Q^2}{(r_1 + r_2)^2 + (s_1 + s_2 - 2pM)^2} \end{aligned}$$

or putting in the values of $P$ and $Q$—

$$\left.\begin{aligned} R &= \frac{(r_1+r_2)(r_1r_2-s_1s_2+p^2M^2)+(s_1+s_2-2pM)(r_1s_2+r_2s_1)}{(r_1+r_2)^2+(s_1+s_2-2pM)^2} \\ S &= \frac{(r_1+r_2)(r_1s_2+r_2s_1)-(s_1+s_2-2pM)(r_1r_2-s_1s_2+p^2M^2)}{(r_1+r_2)^2+(s_1+s_2-2pM)^2} \\ I &= \sqrt{\left\{\frac{(r_1r_2-s_1s_2+p^2M^2)^2+(r_1s_2+r_2s_1)^2}{(r_1+r_2)^2+(s_1+s_2-2pM)^2}\right\}} \end{aligned}\right\} \quad . \quad (29)$$

**49.** We will complete the present chapter by another example of the use of vector algebra, which is both interesting and instructive.

**Two Circuits, one containing Resistance and Self-induction, and the other Resistance and**

**Capacity, are placed in Series between Alternating-current Mains: find the Fall of Potential along the Inductive and Capacity Parts respectively.**

Let $XY$ (Fig. 22) be the circuit containing resistance $r_1$ and

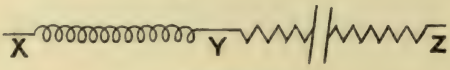

FIG. 22.

self-induction $L$, and $YZ$ that containing resistance $r_2$ and capacity $C$.

Let the R.M.S. value of the P.D. between $X$ and $Z$ be $V$; that between $X$ and $Y$, $v_1$; and that between $Y$ and $Z$, $v_2$.

Let $i$ be the R.M.S. current flowing along $XYZ$.

Put

$$2\pi nL = s_1$$

and—

$$-\frac{1}{2\pi nC} = s_2$$

so that $s_1$ and $s_2$ are the reactances of $XY$ and $YZ$ respectively, and $n$ is the frequency of the current.

Then we have (see § 43), since it is a series arrangement—

$$V = \{(r_1 + r_2) + k(s_1 + s_2)\}i$$
$$v_1 = (r_1 + ks_1)i$$
$$v_2 = (r_2 + ks_2)i$$

whence the **magnitudes** of $V$, $v_1$, and $v_2$ are respectively given by—

$$V = \sqrt{\{(r_1 + r_2)^2 + (s_1 + s_2)^2\}}i$$
$$v_1 = \sqrt{r_1^2 + s_1^2} \,.\, i$$
$$v_2 = \sqrt{r_2^2 + s_2^2} \,.\, i$$

therefore—

$$v_1 : v_2 : V = \sqrt{r_1^2 + s_1^2} : \sqrt{r_2^2 + s_2^2} : \sqrt{\{(r_1 + r_2)^2 + (s_1 + s_2)^2\}}$$

To more fully illustrate this case, suppose that—

$$n = 100$$
$$L = 5 \text{ henrys}$$
$$r_1 = 5 \text{ ohms}$$
$$r_2 = 10 \text{ ohms}$$
$$C = 0{\cdot}5 \text{ microfarad}$$

Then—

$$s_1 = 2\pi \times 100 \times 5 = 3141{\cdot}6$$

$$s_2 = -\frac{2}{2\pi \times 100 \times 10^{-6}} = -3183$$

whence—

$$\sqrt{r_1^2 + s_1^2} = 3142 \text{ nearly}$$

$$\sqrt{r_2^2 + s_2^2} = 3183 \text{ nearly}$$

and—

$$\sqrt{(r_1 + r_2)^2 + (s_1 + s_2)^2} = 44 \text{ nearly}$$

Therefore—

$$v_1 : v_2 : V = 3142 : 3183 : 44 \text{ approximately}$$

That is, if—

$$V = 100 \text{ volts}$$

then—

$$v_1 = 7141 \text{ volts approximately}$$

and—

$$v_2 = 7234 \text{ volts approximately}$$

This shows that, even if low voltage mains are used, dangerous potential differences between portions of reactive circuits may exist.

## PROBLEMS ON CHAPTER VIII.

1. What is the phase difference between the current in, and potential difference between the terminals of, a coil whose resistance is 10 ohms, and self-induction 0·075 henry, when the frequency is 50 periods per second?

*Answer.* 67° nearly.

2. What frequency would make the phase difference 45° with the coil in Question 1?

*Answer.* 21·22 periods per second.

3. What self-induction must a coil whose resistance is 7·5 ohms have in order that, when placed in a circuit of frequency 100, the phase difference between the P.D. between its terminals and the current flowing in it may be 60?

*Answer.* 0·02074 henry.

4. What resistance must be placed in series with a self-induction of 0·025 henry, in a circuit of frequency 75 periods per second, in order that the phase difference between the current and P.D. between the terminals of the arrangement may be 30°?

*Answer.* 20·4 ohms.

5. Calculate the wattless and energy currents in Questions 1, 3, 4, in terms of the total current $i$.

*Answers.*
1. Energy current $= 0{\cdot}39i$;
wattless „ $= 0{\cdot}92i$.
3. Energy current $= 0{\cdot}5i$;
wattless „ $= 0{\cdot}865i$.
4. Energy current $= 0{\cdot}866i$;
wattless „ $= 0{\cdot}5i$.

6. A non-inductive resistance of 100 ohms is placed in series with a resistanceless inductance of 0·05 henry. If a P.D. of 1000 volts is applied between the terminals of the arrangement, how is the fall of potential distributed, the frequency being 100?

*Answer.* Fall of potential along resistance = 954 volts.
„ „ „ inductance = 299·8 „

7. If in Question 6 the inductive part had a resistance of 50 ohms, how would the fall of potential be distributed?

*Answer.* Fall of potential along non-inductive resistance = 652·53 volts.
„ „ „ inductive „ = 384·99 „

8. A non-inductive resistance of 10 ohms is placed in parallel with a resistanceless inductance of 0·025 henry. A P.D. of 100 volts is applied between their common terminals, the frequency being 80. What current flows in the two branches, and what in the main circuit?

*Answer.* Current in non-inductive resistance = 10 amperes.
Current in inductance = 7·958 amperes.
Total current = 12·78 amperes.

9. A non-inductive resistance of 10 ohms is placed in parallel with an inductive resistance whose self induction is 0·025 henry, and resistance 5 ohms. A P.D. of 100 volts is applied between their common terminals, the frequency being 80. What current flows in the two branches, and what is the total current?

*Answer.* Current in non-inductive resistance = 10 amperes.
Current in inductive resistance = 7·394 amperes.
Total current = 14·467 amperes.

10. What is the impedance of the combination in Question 9?

*Answer.* 6·9 ohms.

11. If two coils having self-inductions $L_1$, $L_2$ respectively, and, having negligible resistances, are connected in parallel, what is the self-induction of the combination?

*Answer.* $\dfrac{L_1L_2}{L_1 + L_2}$.

12. If the coils in Question 11 have self-inductions 0·05 and 0·025 respectively, what is the self-induction of the combination?

*Answer.* 0·0167 henry.

13. If two coils having resistances of 1 and 2 ohms respectively, and self-inductions of 0·05 and 0·1 henry respectively, be connected in parallel, what is

(1) the equivalent resistance, (2) the equivalent self-induction, and (3) the impedance of the combination, the frequency being 100 periods per second?

*Answer.* (1) 0·62175 ohms; (2) 0·0333 henry; (3) 21 ohms.

14. An inductance of 0·05 henry having negligible resistance, and a capacity of 3 microfarads, are connected in parallel. A P.D. of 100 volts is applied between their common terminals, and the frequency is 100 periods per second. Find the current in each of the branches and in the main circuit respectively.

*Answer.* Current in inductive branch = 3·1831 amperes.
„ condenser „ = 0·1885 „
Total current = 2·9946 amperes.

15. What capacity is necessary so that, when placed in parallel with an induction of 0·05 henry, there will be no current in the main circuit, the resistances of the circuits being negligible, and the frequency being 75 periods per second?

*Answer.* 90 microfarads.

16. Two circuits, one having a resistance of 10 ohms, and a self-induction of 0·125 henry, and the other having a resistance of 100 ohms, and a capacity of 25 microfarads, are placed in parallel between the terminals of an alternator giving 1000 volts at a frequency of 50 periods per second: what are the currents in the two branches, and in the main circuit?

*Answer.* Current in inductive branch = 24·94 amperes.
„ condenser „ = 6·17 „
Total current = 21·41 „

17. What is the impedance of the combination in Question 16?

*Answer.* 46·7 ohms.

18. What is the self-induction of a coil of negligible resistance such that, when placed in series with a non-inductive apparatus taking 10 amperes at 40 volts, will allow of the arrangement being used on a 100-volt 50-frequency circuit?

*Answer.* 0·029 henry.

19. If in Problem 18 the apparatus had a resistance of 2 ohms, and a self-induction of 0·011 henry, what should be the self-induction of the coil?

*Answer.* 0·02015 henry.

20. A coil whose self-induction is 0·04 henry is used in series with an arc lamp which takes 7·5 amperes at 40 volts, the whole being in a 100-volt circuit: what is the frequency of the current?

*Answer.* 43·3 periods per second.

# CHAPTER IX.

## The Use of Sine Curves in Alternating-current Problems. Effect of Higher Harmonics.

**50.** The representation of alternating currents and E.M.F.s by sine, or simple periodic, curves is frequently objected to on the ground that they do not accurately represent the actual variations of the current or E.M.F., as the case may be, and consequently cannot lead to accurate results.

Let us examine carefully the value of this objection, and ascertain whether we may expect to obtain true results when we assume that alternating currents and E.M.F.s may be expressed as sine functions of the time.

It will readily be granted that all alternating currents and E.M.F.s are periodic; that is, that they are all of such a nature that there is a certain time, $T$, called the periodic time, in which their values go through a complete cycle of changes, and that in each succeeding time $T$ this cycle is repeated. It is quite correct, then, to represent any alternating current or E.M.F. whatever by some **periodic** function of the time.

Now, by a theorem due to Fourier, any periodic function of the time of frequency $n$ may be represented by an expression of the form—

$$a_1 \sin(pt - \theta_1) + a_2 \sin(2pt - \theta_2) + a_3 \sin(3pt - \theta_3) + \ . \ . \ . \text{ etc.}$$

where $p = 2\pi n$, and $a_1$, $a_2$, etc., are the amplitudes, and $\theta_1$, $\theta_2$, etc., the phases.

The frequency of the first term is $n$, and those of the other terms are respectively $2n$, $3n$, etc. These terms are called the first, second, etc., **harmonics** of the first term, which is itself called the **fundamental** term.

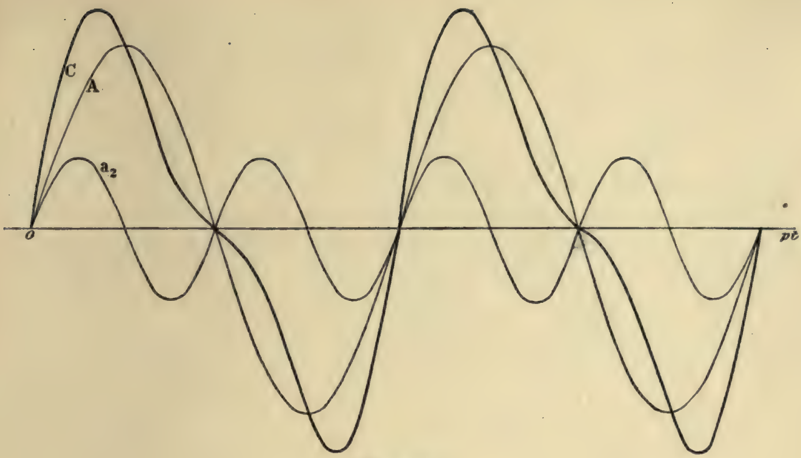


FIG. 23.

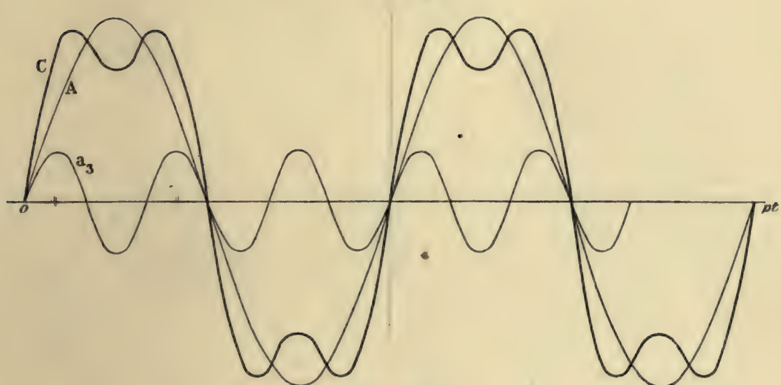


FIG. 24.

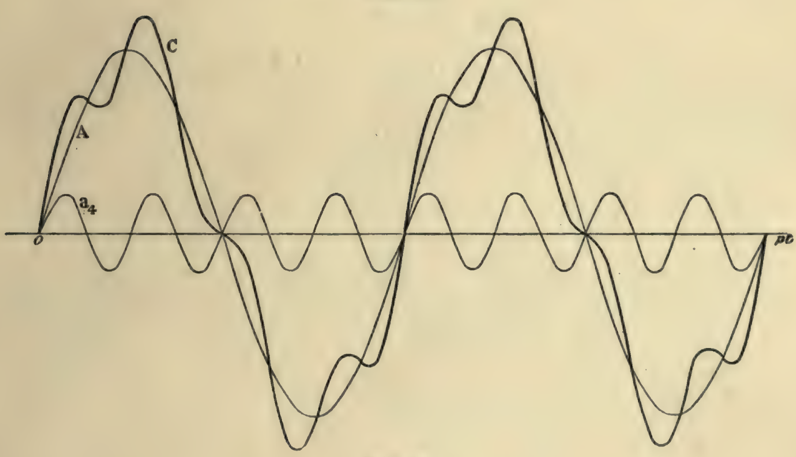


FIG. 25.

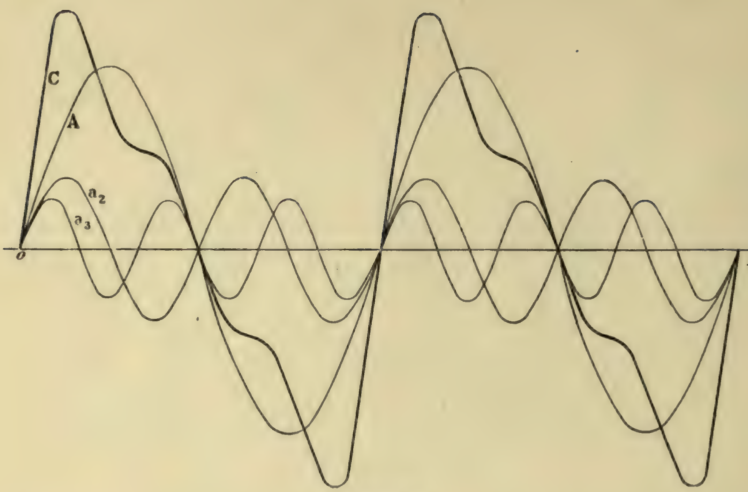


FIG. 26.

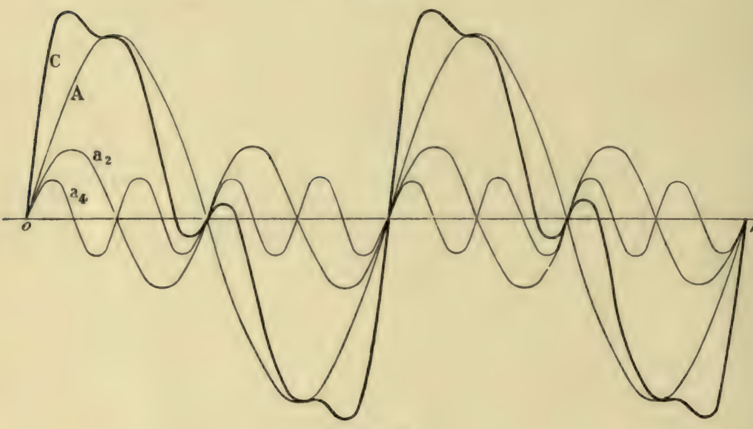


FIG. 27.

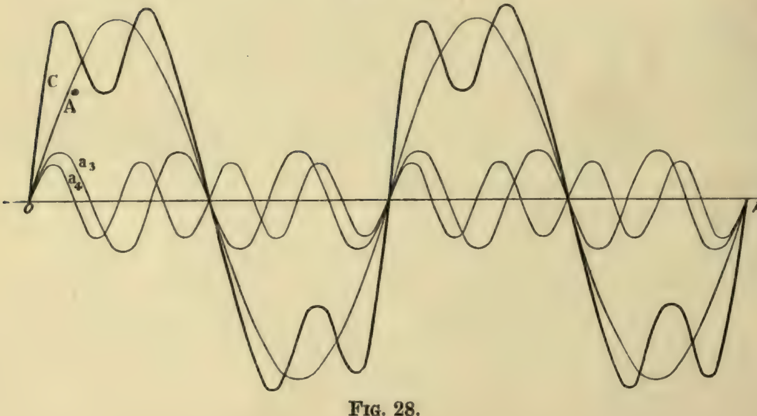


FIG. 28.

Fig. 23 shows a curve $C$ compounded of the fundamental and the first harmonic; in Fig. 24 the curve $C$ is compounded of the fundamental and the second harmonic; in Fig. 25 the curve $C$ is compounded of the fundamental and the third harmonic; in Fig. 26 the curve $C$ is compounded of the fundamental and the first and second harmonics; in Fig. 27 the first and third harmonics are present; in Fig. 28 the second and third harmonics are present; whilst in Fig. 29 the curve $C$ is compounded of the fundamental and the first, second, and third harmonics. In each case $A$ is the fundamental curve, $a_2$, $a_3$, $a_4$, the first, second, and third harmonics respectively, and $C$ the resultant curve.

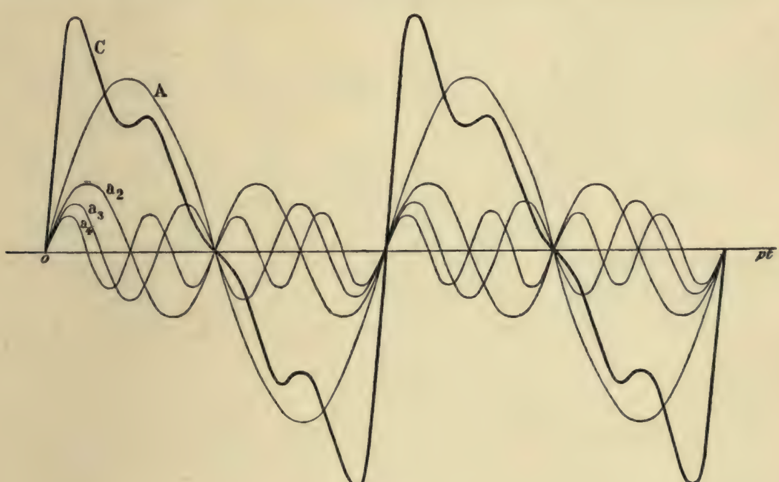


FIG. 29.

The first, third, etc., harmonics are called the even harmonics, because their periodic times are even submultiples of that of the fundamental. The second, fourth, etc., harmonics are, for like reason, called the odd harmonics.

The reader should practise drawing curves compounded of the fundamental and harmonics, as they are very instructive.

**51.** It is a matter of experience that the even harmonics are generally absent from the curves representing alternating currents and E.M.F.s, so that we may legitimately represent them by expressions of the form—

$$a_1 \sin (pt - \theta_1) + a_3 \sin (3pt - \theta_3) + a_5 \sin (5pt - \theta_5) + \ . \ . \ . \text{ etc.}$$

It may be that, in a transformer or other induction machine, either the current or E.M.F. follows the simple sine law. The presence of iron, however, causes one of them to deviate from the sine law. If a sine potential difference is applied between the terminals of the primary of a transformer, the primary-current wave will be compounded of a sine function and some of its higher odd harmonics, and will consequently be distorted.

If a sine current is produced, the applied potential difference is distorted.

## 52. Root Mean Square Value of a Curve compounded of a Fundamental Sine Curve and its Higher Odd Harmonics.

Let—

$$y = a_1 \sin (pt - \theta_1) + a_3 \sin (3pt - \theta_3) + \ldots \text{ etc.}$$

then—

$$\begin{aligned} y^2 = a_1^2 \sin^2 (pt - \theta_1) &+ a_3^2 \sin^2 (3pt - \theta_3) + \ldots \text{ etc.} \\ &+ 2a_1 a_3 \sin (pt - \theta_1) \sin (3pt - \theta_3) + \ldots \text{ etc.} \\ &+ 2a_r a_s \sin (rpt - \theta_r) \sin (spt - \theta_s) + \ldots \text{ etc.} \end{aligned}$$

The mean square value of $y$ is—

$$\frac{p}{2\pi}\int_0^{\frac{2\pi}{p}} y^2 dt$$

that is, the mean square value of $y$ is the sum of a series of terms, such as—

$$\frac{p}{2\pi}\int_0^{\frac{2\pi}{p}} a_r^2 \sin^2 (rpt - \theta_r) dt$$

together with the sum of a series of terms, such as—

$$\frac{p}{2\pi}\int_0^{\frac{2\pi}{p}} 2a_r a_s \sin (rpt - \theta_r) \sin (spt - \theta_s) dt$$

where $r$ and $s$ are always ~~old~~ odd integers.

Now—

$$\frac{p}{2\pi}\int_0^{\frac{2\pi}{p}} a_r^2 \sin^2 (rpt - \theta_r) dt = \frac{a_r^2}{2}$$

if $r$ is any integer whatever, and—

$$\frac{p}{2\pi}\int_0^{\frac{2\pi}{p}} 2a_r a_s \sin(rpt - \theta_r)\sin(spt - \theta_s)dt = 0$$

if $r$ and $s$ are either both even or both odd. But as we have only to consider the case in which $r$ and $s$ are always odd, the mean square value of $y$ is—

$$\frac{a_1^2 + a_3^2 + a_5^2 + \ldots \text{ etc.}}{2}$$

therefore the R.M.S. value of $y$ is—

$$\sqrt{\frac{a_1^2 + a_3^2 + a_5^2 + \ldots \text{ etc.}}{2}}$$

which is independent of $\theta_1$, $\theta_2$, $\theta_3$, etc.; that is, the R.M.S. values of functions containing only the fundamental and odd (or even, but not both) harmonics depend solely on their maxima values, and are independent of their relative phases.

The R.M.S. value of any alternating current or E.M.F. can therefore be represented by a definite vector.

### 53. Root Mean Square Values of Alternating Currents and E.M.F.s can be compounded as Vectors, whether they be Simple Sine Functions or not.

We now proceed to show that all R.M.S. values of alternating currents and E.M.F.s can be compounded as vectors, whatever be the shape of the curve representing them.

In Fig. 30, let $AB$ be a non-inductive resistance $R$, and $BC$ a coil having resistance and self-induction.

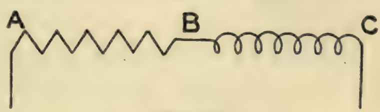

FIG. 30.

Let $v_1$, $v_2$, and $v$ be the corresponding instantaneous values of the P.D. between $A$ and $B$, $B$ and $C$, $A$ and $C$ respectively, and let the respective R.M.S. values be $V_1$, $V_2$, and $V$.

Let $i$ be the instantaneous current passing through the circuit, and $P$ the power given to the inductive circuit $BC$.

Then—

$$v = v_1 + v_2$$

therefore—

$$v^2 = v_1^2 + 2v_1v_2 + v_2^2$$

and by Ohm's law—

$$= v_1^2 + 2Riv_2 + v_2^2$$

therefore—

$$2Riv_2 = v^2 - v_1^2 - v_2$$

therefore—

$$\frac{2R}{T}\int_0^T iv_2 dt = \frac{1}{T}\int_0^T (v^2 - v_1^2 - v_2^2)dt$$

where $T$ is the periodic time.

Therefore—

$$2RP = V^2 - V_1^2 - V_2^2$$

or—

$$P = \frac{V^2 - V_1^2 - V_2^2}{2R}$$

In proving this formula (which gives the three-voltmeter method of measuring the power given to an inductive circuit), no assumption whatever is made respecting the shape of the E.M.F. curves.

Now, let $I$ be the R.M.S. current flowing through $ABC$; then, since $R$ is non-inductive, we have—

$$R = \frac{V_1}{I}$$

therefore—

$$P = I \, . \, \frac{V^2 - V_1^2 - V_2^2}{2V_1}$$

$$= IV_2 \, . \, \frac{V^2 - V_1^2 - V_2^2}{2V_1V_2}$$

If we now put—

$$\frac{V^2 - V_1^2 - V_2^2}{2V_1V_2} = \cos\phi$$

which is admissible, since $v$ is never greater than $v_1 + v_2$, we get—

$$P = V_2 I \cos\phi$$

which is the same as if the quantities were simple sine functions of the time, and $\phi$ the angle of lag or lead.

The point to notice is that the expression for $\cos\phi$ is what it would be if the P.D.s were definite vectors. This is seen by reference to Fig. 31.

We may call $\phi$ the **equivalent** phase-difference between

$V_2$ and $I$. To give a geometrical interpretation, $\phi$ is the phase-difference between two simple periodic functions whose R.M.S. values are $V_2$ and $I$.

We call an **equivalent sine curve** one which has the same R.M.S. value as a given periodic curve which is not itself necessarily a simple sine curve.

**54.** It follows that, so long as the E.M.F.s and currents are such that each half-wave is identical with the preceding one, except in sign, as is usually the case, their R.M.S. values can be compounded in a vector fashion.

If we are dealing with a circuit whose self-induction and resistance are constant, the diagram of E.M.F.s is shown in Fig. 32.

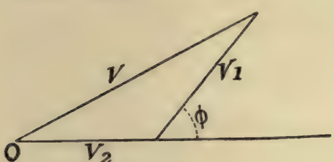


FIG. 31.

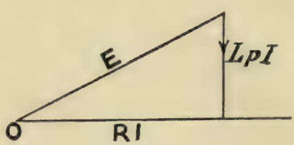


FIG. 32.

$E$ is the R.M.S. value of the impressed P.D. The E.M.F. necessary to overcome the resistance is $RI$, and that required to balance the self-induction is $pLI$, where $p = 2\pi$ times the frequency, $L$ is the self-induction, and $I$ the R.M.S. value of the current.

If there is iron in the immediate neighbourhood, the permeability depends upon the current, and the self-induction of the circuit is no longer a constant quantity.

In consequence of this the E.M.F. curve becomes distorted, and differs in shape from the current curve.

It should be noticed that Fig. 32 holds good either for maxima or R.M.S. values of E.M.F. and current, when the self-induction of the circuit is constant, and the impressed P.D. is a sine function of the time.

We have seen that the power given to an inductive circuit may be written in the form of—

$$P = VI \cos \phi$$

where $V$ and $I$ are R.M.S. values, and $\cos \phi$ is a multiplying factor less than unity. This shows that in cases where only R.M.S. values are concerned, the effects are just the same as if the actual P.D. and current were replaced by a sine P.D. and current having

**the same R.M.S. values, and having a difference of phase equal to $\phi$.**

**55.** We are now in a position to say that calculations based on the assumption that the E.M.F.s and currents are simple sine functions are perfectly legitimate, provided (1) that only R.M.S. values are involved, (2) that the assumed sine values of the E.M.F.s and currents have the same R.M.S. values as the actual E.M.F.s and currents have, and (3) that only odd terms of Fourier's Series are involved.

We can, for instance, calculate the effects of hysteresis and eddy currents on the assumption of equivalent simple periodic values, but we cannot, on the same assumption, draw any conclusion whatever respecting the amount of insulation necessary in any particular case, since the maximum E.M.F., and not its R.M.S. value, determines the insulation needed.

## PROBLEMS ON CHAPTER IX.

1. What are the maxima values of the equivalent sine curves of the following?—

(i.) $1000 \sin 750t + 100 \sin 2250t$.
(ii.) $100 \sin 300t + 10 \sin 900t + 5 \sin 1500t$.
(iii.) $250 \sin 240t + 50 \sin 720t + 10 \sin 120\ t$.

*Answers.* (i.) 1005; (ii.) 100·62; (iii.) 255·1.

2. What are the equivalent phase differences between the following pairs of curves?—

$$\text{(i.)}\ \begin{cases} i = 100 \sin 300t \\ e = 500 \sin\left(300t - \frac{\pi}{6}\right) + 50 \sin\left(900t - \frac{\pi}{6}\right) \end{cases}$$

$$\text{(ii.)}\ \begin{cases} i = 100 \sin 450t + 10 \sin 1350t \\ e = 300 \sin\left(450t - \frac{\pi}{4}\right) + 30 \sin\left(1350t - \frac{\pi}{4}\right) \end{cases}$$

*Answers.* (i.) 30° 30′. (ii.) 45°.

# CHAPTER X.

Choking Coils for Non-inductive and Inductive Circuits—Design of Choking Coils.

## Impedance Coils.

**56.** Impedance coils, or choking coils, as they are often called, are simply coils having low ohmic resistance and high self-induction. They are formed by winding a coil of copper wire round a laminated iron core, and are used for the purpose of absorbing a portion of the pressure between constant potential alternating current mains when it is desired to place between the mains any apparatus which requires a voltage less than that between the mains.

For example, a single open arc lamp requires a pressure of about 40 volts, and if such a lamp is to be run from 100-volt mains, the pressure must be by some means lowered to suit its requirements.

An obvious means of accomplishing this would be to place a non-inductive resistance in series with the lamp such that about 60 volts would be required to drive the current against the resistance, but this is an extremely wasteful device. To illustrate the point, suppose that the arc lamp required a current of 10 amperes at a pressure of 40 volts, and that only 100 volts was available. A resistance of 6 ohms placed in series would then absorb 60 volts, leaving the necessary 40 volts to run the lamp. The power used in running the lamp would be 400 watts, while that wasted in the resistance would be 600 watts. If, as is often the case, the consumer was charged so much per ampere-hour, the waste energy does not affect him, since in either case he pays too much for the energy he uses; if, on the other hand, he is charged per kilowatt-hour, he has to pay for a single lamp for as much energy as would

work $2\frac{1}{2}$ lamps. Thus if an ampere-hour meter, or coulomb meter, is used, the consumer is the loser; and if a watt-hour meter, or energy meter, is employed, the consumer does not get value for his money. The use of a series resistance is, therefore, inadmissible.

With continuous-current mains, the only alternative to a resistance is a motor-generator; but with an alternating-current supply, the reduced voltage may be otherwise obtained: (1) by the use of a choking coil, (2) by means of a step-down transformer, and (3) by a motor-generator of some description.

Of these methods the use of a step-down transformer, or a choking-coil, is generally adopted. Either of these is equally satisfactory to the consumer, provided the primary coil of the transformer is only excited when required. The supplier prefers to use transformers because the inductance is small on closed secondary circuits, whereas if many choking coils are connected with the supply mains the load will be highly inductive, and the regulation of the generating plant will be rendered very difficult (see Chap. XIII.).

**57.** There are two cases for consideration: (1) that in which the apparatus to be worked is non-inductive, and (2) that in which it is inductive.

CASE 1. **Apparatus Non-inductive.**

Let the R.M.S. voltage of the supply be $V$.

Let $v_1$ and $i$ be the pressure and current required for working the non-inductive apparatus.

Let $v$ be the pressure between the terminals of the choking coil necessary for the purpose.

We shall assume that the resistance of the choking coil is negligible, so that the current $i$ lags 90° behind the P.D. between its terminals. Also, the P.D. $v_1$ is in phase with $i$, since the apparatus is non-inductive.

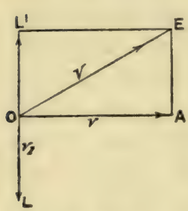


FIG. 33.

Let $OA$ (Fig. 33) represent $v_1$, and $OL$ represent $v$.

Then $V$ has to supply a component $OA$ to drive the current against the resistance of the apparatus, and another component $OL'$ to balance the E.M.F. of self-induction of the choking coil; that is, $V$ is represented by the vector $OE$.

We therefore have—

$$V^2 = v^2 + v_1^2$$

or—

$$v^2 = V^2 - v_1^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If, therefore, $L$ is the self-induction of the choking coil, and $\cancel{p} = 2\pi\cancel{n}$, where $n$ is the frequency of supply, we have—
$\omega$ $f$

$$pLi = v$$
$$= \sqrt{V^2 - v_1^2}$$

Therefore, the self-induction of the choking coil necessary for the purpose is given by—

$$L = \frac{1}{pi}\sqrt{V^2 - v_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$i$ being the R.M.S. value of the current in amperes.

CASE 2. **Apparatus Inductive.**

Let the notation be the same as in Case 1.

Let, in addition, the resistance of the apparatus be $r_1$, and its self-induction $L_1$.

Then the current $i$ lags behind $v_1$ by an angle $\theta$, where (see § 21)—

$$\tan\theta = \frac{pL_1}{r_1}$$

Let $OA$ (Fig. 34) represent $v_1$, and $OI$ represent $i$, where the angle $AOI = \theta$.

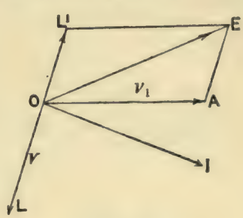


FIG. 34.

Then $OL$, which represents the E.M.F. of self-induction of the choking coil, is 90° behind $OI$. The P.D. $V$ has to supply a component equal to $OA$ to work the apparatus, and another equal to $OL'$ to balance the self-induction of the choking coil, and is therefore given by $OE$.

Now, the angle $AOL' = \frac{\pi}{2} - \theta$; therefore—

$$V^2 = v^2 + v_1^2 + 2vv_1\cos\left(\frac{\pi}{2} - \theta\right)$$
$$= v^2 + v_1^2 + 2vv_1\sin\theta$$

whence—

$$v = \sqrt{V^2 - v_1^2\cos^2\theta} - v_1\sin\theta \quad . \quad . \quad . \quad (3)$$

and the self-induction $L$ of the choking coil is given by—

$$L = \frac{1}{pi}\{\sqrt{V^2 - v_1^2 \cos^2 \theta} - v_1 \sin \theta\} \quad . \quad . \quad . \quad (4)$$

If $\theta = 0$, equation (4) reduces to equation (2), as should be the case, since that is the condition that the apparatus should be non-inductive.

As a rule, the value of $\theta$ will be known for any particular class of apparatus, such as induction motors, so that $V$, $v_1$, and $\theta$ are all given, and $L$ becomes determinate.

## Design of Choking Coils.

**58.** Having determined the value of $L$ for the choking coil required, we proceed to indicate the principles upon which its design and construction may be carried out.

Let $N$ be the number of turns of wire on the coil, and $F \sin pt$ the flux of magnetic lines through the iron core at time $t$.

Then the E.M.F. produced in the coil by the variation of this flux is given by—

$$e = \frac{d}{dt}(NF \sin pt)$$
$$= pNF \cos pt$$

The R.M.S. value of $e$ is therefore—

$$\frac{pNF}{\sqrt{2}}$$

or, reduced to volts, we have—

$$v = \frac{pNF}{10^8\sqrt{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

therefore—

$$\frac{pNF}{10^8\sqrt{2}} = pLi$$

or—

$$NF = 10^8\sqrt{2}Li \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Thus the product of the number of turns of wire, and the maximum magnetic flux is given by equation (6).

Now, let $A$ be the area in square centimetres of the cross section of the iron core, and $B$ the maximum induction in the core, so that—

$$F = AB$$

and—

$$NF = NAB$$

The maximum induction $B$ may be fixed arbitrarily, thus giving the product $NA$. It is not necessary to keep the value of $B$ so low as in transformer cores, because a choking coil is only in use while the accompanying apparatus is working, and hysteresis and eddy-current losses are not of such great importance as if, as is often the case in transformers, the current passed through it constantly, and the iron loss was always going on.

The determination of $N$ and $A$ separately is a matter of design, and is mainly a question of experience.

We will complete the calculation in a specific case.

A choking coil is required for use with a 40-volt 5-ampere arc lamp on a 100-volt 50 frequency supply: give the necessary details of the coil.

The voltage $v$ required between the terminals of the coil is—

$$\begin{aligned} v &= \sqrt{100^2 - 40^2} \\ &= 91{\cdot}6 \text{ volts, nearly} \end{aligned}$$

therefore—

$$\frac{pNF}{10^8\sqrt{2}} = 91{\cdot}6$$

whence—

$$NF = 41{\cdot}23 \times 10^6$$

Suppose that—

$$B = 14{,}000$$

then—

$$NA = 2943$$

Now, if the iron core forms a closed circuit, so as to have a small reluctance, and if we take

$$N = 100$$

we have—

$$A = 29{\cdot}43 \text{ square centimetres.}$$

Since $B = 14{,}000$, the permeability is 823.

With a current of 5 amperes, the magnetizing force is—

$$\frac{4\pi \times 100 \times 5}{10l} = \frac{628 \cdot 32}{l}$$

where $l$ is the length of the magnetic circuit, *i.e.* the average length of the closed lines of magnetic induction.

The induction is therefore—

$$823 \times \frac{628 \cdot 32}{l}$$

Therefore—

$$823 \times \frac{628 \cdot 32}{l} = 14{,}000$$

whence—

$$l = 37 \text{ centimetres}$$

so that the core could be built up of iron stampings of section shown in Fig. 35 to a thickness of 9·8 centimetres.

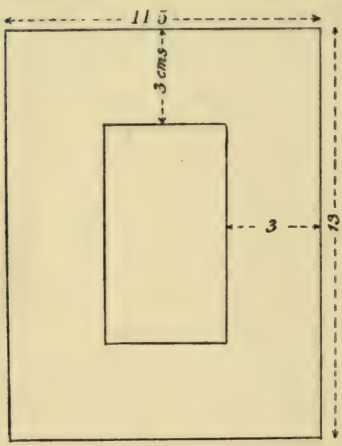


Fig. 35.

The wire to be used should be equivalent to No. 14 S.W.G., since this will carry 5 amperes at a current density of 1000 amperes per square inch.

We may notice that the self-induction of the choking coil is given by—

$$pLi = v$$
$$= 91 \cdot 6$$

therefore—

$$L = \frac{91 \cdot 6}{2\pi \times 50 \times 5}$$
$$= 0 \cdot 0583 \text{ henry}$$

The foregoing calculation is not quite correct, since we have omitted to take into account the iron losses of the core of the choking coil. These could not be allowed for until the dimensions of the core were fixed.

The following table gives the hysteresis loss in ergs per cycle per cubic centimetre of good soft annealed iron :—

TABLE II.

HYSTERESIS LOSS.

| Maximum induction per square centimetre. | Ergs lost per cycle per cubic centimetre. |
|---|---|
| 2,000 | 420 |
| 3,000 | 800 |
| 4,000 | 1,230 |
| 5,000 | 1,700 |
| 6,000 | 2,200 |
| 7,000 | 2,760 |
| 8,000 | 3,450 |
| 9,000 | 4,200 |
| 10,000 | 5,000 |
| 11,000 | 5,820 |
| 12,000 | 6,720 |
| 13,000 | 7,650 |
| 14,000 | 8,650 |
| 15,000 | 9,670 |

Since the induction in the core is 14,000 lines per square centimetre, the hysteresis loss in ergs per cubic centimetre per cycle is 8650.

The volume of the iron is 1089 c.cms.; therefore the total loss in ergs per cycle is—

9,419,850

To provide for this requires—

47·1 watts

and takes a current of—

0·51 ampere at 91·6 volts.

As this current is only 9 per cent. of the total current, the loss of energy in the choking coil is but small.

The same remark applies for the energy current required for the eddy-current losses.

By working at a lower induction in the iron, and by making the magnetic circuit open instead of closed, the losses may be still further diminished.

## PROBLEMS ON CHAPTER X.

1. What is the self-induction of a coil of negligible resistance that, when placed in series with a non-inductive apparatus taking 10 amperes at 40 volts, will allow the arrangement to be used on a 100-volt 50-frequency circuit?

*Answer.* 0·029 henry.

2. If in Question 1 the apparatus had a resistance of 2 ohms, and a self-induction of 0·011 henry, what must be the self-induction of the coil?

*Answer.* 0·02015 henry.

3. Give a complete design of a choking coil with open magnetic circuit to give a counter E.M.F. of 90 volts, and taking 25 amperes on a 75-frequency circuit.

4. A choking coil of negligible resistance absorbs 90 volts on a circuit of frequency 100: what is its counter E.M.F., if the frequency is 75, the current remaining the same?

*Answer.* 67·5 volts.

5. A choking coil of negligible resistance gives the same counter E.M.F. at frequencies 100 and 80: what is the ratio of the currents at the two frequencies?

*Answer.* $\frac{4}{5}$.

6. A choking coil has a P.D. between its terminals of 90 volts; its resistance is 0·5 ohm, and the current is 20 amperes: what is the self-induction of the coil, and what is the difference of phase between the current and P.D., the frequency being 75?

*Answer.* Self-induction = 0·00955 henry
phase-difference = 83° 40′.

7. If a P.D. represented by—

$$e = 1000 \sin 750t + 100 \sin 2250t$$

is applied between the terminals of a coil whose resistance is 10 ohms, and self-induction 0·05 henry, find the power given to the circuit.

*Answer.* 3354 watts.

# CHAPTER XI.

Transformer on Open Secondary Circuit—Hysteresis Loss—Eddy-current Loss—Lamination of Core—Transformer working under Load—Calculation of Eddy-current Loss—Effect of Shape of P.D. Curve on Iron Losses—Tests of Transformers.

## MONO-PHASE TRANSFORMERS.

**59.** A mono-phase stationary transformer consists, in its simplest form, of two coils of wire wound on the same iron core, as in Fig. 36.

Between the terminals of one of the circuits—the primary circuit—an alternating P.D. is applied. This causes a current to flow through the primary coil, producing an alternating magnetic field which cuts the windings of the other circuit—the secondary circuit. An alternating E.M.F. is thus induced in the secondary circuit, giving rise to a secondary current if the secondary terminals are connected to an external circuit.

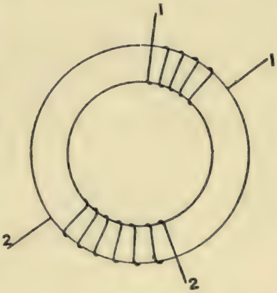

FIG. 36.

**Transformer on Open Secondary Circuit.** — It will readily be seen that the presence of a secondary circuit renders a study of the transformer somewhat complicated, since the mutual reactions of the two currents have to be taken into consideration.

To make matters as simple as possible, it is advisable to commence by supposing that the secondary circuit is open, so as to have only the primary current and its effects to consider.

Let $E_1$ be the R.M.S. potential difference applied between the primary terminals, $r_1$ the resistance of the primary coil, $s_1$ its reactance, and $i_1$ the R.M.S. value of the primary current on open secondary circuit.

Then—

$$i_1 = \frac{E_1}{\sqrt{r_1^2 + s_1^2}}$$

Since the circuit is wound on an iron core, $s_1$ will be large, and $i_1$ correspondingly small.

If the power factor of the circuit is now determined, it will be found to be comparatively small, showing that a considerable portion of the current is at right angles to the applied P.D., and is **idle,** or **wattless.** There will thus be an energy component of the current in phase with the P.D., and a wattless component at right angles to the P.D.

We have to indicate the functions of these two components of the primary current when the secondary circuit is open.

**Hysteresis Loss.**—It is known that when iron is subjected to a complete cycle of magnetic changes a certain amount of energy is wasted due to hysteresis. The exact amount of energy thus lost per cycle depends upon the quality of the iron and the maximum induction in it (see Chap. X., Table II.).

If, further, the cycle is such that the magnetizing force varies continuously from zero to a positive maximum, back again through zero to an equal negative maximum, and finally to zero again, the energy lost per cycle due to hysteresis may be represented by the formula—

$$H = kB^{1\cdot4 \text{ to } 1\cdot6} \text{ (Steinmetz's law)}$$

where $H$ is the energy lost, and $k$ is a constant depending upon the quality of the iron, and $B$ is the maximum value of the induction.

It should here be mentioned that Steinmetz's formula holds good only for somewhat low induction densities, such as are practically met with in transformers. As the induction density increases, the exponent of $B$ diminishes.

**Eddy-current Loss.**—Another source of waste energy is **eddy currents.**

The iron core being itself a conductor of electricity, secondary currents are induced in it. These currents flow in the mass of the iron in closed circuits, approximately in planes at right angles to the magnetic flux. The result is that the core becomes heated, and energy is dissipated.

**Lamination of Core.**—To reduce these eddy-current losses as much as possible the core is built up of thin sheets of

iron, insulated from one another by means of thin paper or varnish. Sometimes the oxidization on the surfaces of the iron sheets is of itself sufficient insulation. The core is laminated by planes parallel to the magnetic flux. By this means the eddy currents are confined to narrow paths, which greatly reduces them, and minimizes the consequent loss of energy.

The losses due to eddy currents will be more completely dealt with later on.

When working on open secondary circuit the power given to the primary is equal to the sum of the losses due to hysteresis and eddy currents, together with an insignificantly small $i_1^2 r_1$ loss, which may be neglected. We shall see later that the iron losses are independent of the secondary load.

The function of the wattless component of the primary current is to magnetize the iron. No permanent expenditure of energy is required for this purpose. The magnitude of the wattless component depends upon the permeability of the iron and the reluctance of the magnetic path. It is greater in open than in closed magnetic-circuit transformers.

Some writers on the subject call the total primary current, on open secondary circuit, the magnetizing current; but it is better, in our opinion, to call the **wattless component only** the magnetizing current.

The power component is called the **hysteretic energy current.**

Denoting the magnetizing current and the hysteretic energy currents by $\dot{i}_m$ and $\dot{i}_h$ respectively, and the total current by $\dot{i}_0$, we have—

$$\dot{i}_0^2 = \dot{i}_h^2 + \dot{i}_m^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

since $\dot{i}_h$ and $\dot{i}_m$ are at right angles to each other.

Let $E_0'$ be the R.M.S. value of the E.M.F. required to balance the primary counter E.M.F., and write—

$$\rho = \frac{\dot{i}_h}{E_0'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\sigma = \frac{\dot{i}_m}{E_0'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and—

$$Y = \frac{\dot{i}_0}{E_0'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Then—

$$\rho^2 + \sigma^2 = \frac{i_h^2 + i_m^2}{E_0'^2}$$

$$= \frac{i_0^2}{E_0'^2} = Y^2$$

therefore—

$$Y = \sqrt{\rho^2 + \sigma^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$\rho$ is called the virtual conductance, $\sigma$ the susceptance, and $Y$ the admittance of the primary circuit. The power factor on open secondary circuit is given by (see Fig. 37)—

$$\cos \phi = \frac{i_h}{i_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and should be as small as possible.

When the transformer is on open secondary circuit, the primary current is so small that the loss of energy due to ohmic resistance may be neglected. As soon as load is put on the secondary, the primary current increases, and the copper losses in both circuits must be taken into account.

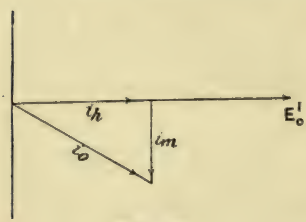

FIG. 37.

In designing a transformer, the maximum induction $B$ in the core is usually fixed beforehand, and the permeability $\mu$ of the iron is therefore known. The value of the magnetizing current $i_m$ may then be determined as follows:—

$$B = \mu H$$

$$= \mu \cdot \frac{4\pi N_1 i_m \sqrt{2}}{10L}$$

where $N_1$ is the total number of primary turns and $L$ the mean length of the magnetic circuit; hence—

$$i_m = \frac{10BL}{4\pi\mu N_1\sqrt{2}}$$

$$= 0{\cdot}5627\frac{BL}{\mu N_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The hysteretic-energy current can also be determined in the following manner:—

Let $V$ be the volume of the iron core in cubic centimetres,

$W$ the number of ergs per cycle lost in hysteresis, $n$ the frequency of the supply, and $E_0$ the primary applied P.D. Then (neglecting eddy currents)—

$$i_h E_0 = \frac{nWV}{10^7}$$

or—

$$i_h = \frac{nWV}{10^7 E_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

We will show later (see § 61) how the loss due to eddy currents may be approximately calculated. The current necessary to supply energy for this loss must be included in $i_h$.

## Transformer Working under Load.

**60.** We now pass on to consider the working of a transformer under load. If a transformer is used to supply power for incandescent lamps only, the external circuit may be treated as non-inductive; if, however, it supplies arc lamps requiring choking coils, or induction motors, the load is inductive, and the external reactance must be taken into account.

We shall assume that the maximum induction in the iron core is constant at all loads.

The method of vector algebra supplies a concise means of performing the necessary calculations.

We shall for the present assume that both the reactances of the two circuits and their mutual induction are constant.

Let the resistance of the primary circuit be $r_1$, its reactance $s_1$, and the R.M S. primary current $i_1$, and let the corresponding quantities for the secondary circuit (internal and external) be $r_2$, $s_2$, and $i_2$, the coefficient of mutual induction between the two circuits $M$, and the R.M.S. potential difference between the primary terminals $e$.

Then, following the argument of § 47, Chap. VIII., the vector equation of E.M.F.s in the primary circuit is—

$$r_1 i_1 + k s_1 i_1 + kpMi_2 = e \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The E.M.F. in the secondary circuit due to mutual induction is $-kpMi_1$. This has to furnish a component, $r_2 i_2$, in phase with the secondary current to drive the current against the ohmic resistance of the circuit, and also a component, $ks_2 i_2$, to balance the

E.M.F., $-ks_2i_2$, due to the secondary reactance. We thus have the vector E.M.F. equation—

$$r_2i_2 + ks_2i_2 = -kpMi_1$$

or—

$$r_2i_2 + ks_2i_2 + kpMi_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Equations (9) and (10) are the vector E.M.F. equations of the primary and secondary circuits respectively. Eliminating first $i_2$ and then $i_1$ we get—

$$\{(r_1r_2 - s_1s_2 + p^2M^2) + k(r_1 + s_2 r_2s_1)\}i_1 = (r_2 + ks_2)e$$

and—

$$\{(r_1r_2 - s_1s_2 + p^2M^2) + k(r_1s_2 + r_2s_1)\}i_2 = -kpMe$$

which may be written in the forms—

$$[\{r_1(r_2^2 + s_2^2) + p^2r_2M^2\} + k\{s_1(r_2^2 + s_2^2) - p^2s_2M^2\}]i_1$$
$$= (r_2^2 + s_2^2)e \quad . \quad (11)$$

and—

$$\{-(r_1s_2 + r_2s_1) + k(r_1r_2 - s_1s_2 + p^2M^2)\}i_2 = pMe \quad . \quad . \quad . \quad (12)$$

Hence the **magnitudes** of the primary and secondary currents are given by—

$$\left.\begin{aligned} i_1 &= \frac{e}{\sqrt{\left\{r_1^2 + r_2^2 + 2p^2M^2\dfrac{(r_1r_2 - s_1s_2)}{r_2^2 + s_2^2} + \dfrac{p^4M^4}{r_2^2 + s_2^2}\right\}}} \\ i_2 &= \frac{pMe}{\sqrt{\{(r_1^2 + s_1^2)(r_2^2 + s_2^2) + 2p^2M^2(r_1r_2 - s_1s_2) + p^4M^4\}}} \end{aligned}\right\} . \ (13)$$

If $R$ and $S$ are the equivalent resistance and reactance of the primary circuit, we get at once from equation (11)—

$$\left.\begin{aligned} R &= r_1 + \frac{p^2r_2M^2}{r_2^2 + s_2^2} \\ \text{and—} \quad S &= s_1 - \frac{p^2s_2M^2}{r_2^2 + s_2^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

From equation (11) we also see that the primary current lags behind the applied P.D. by an angle $\theta$ where—

$$\tan\theta = \frac{s_1(r_2^2 + s_2^2) - p^2s_2M^2}{r_1(r_2^2 + s_2^2) + p^2r_2M^2} \quad . \quad . \quad . \quad . \quad (15)$$

Whence the current and applied P.D. are in phase—that is, there is electrical resonance in the primary circuit, if—

$$s_1 s_2^2 - p^2 s_2 M^2 + s_1 r_2^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (16)$$

that is, for a given primary reactance there are two values of the secondary reactance for which resonance may occur in the primary circuit, provided the roots of equation (16) considered as a quadratic in $s_2$ are real; that is, if—

$$p^2 M^2 \text{ is greater than } 2s_1 r_2$$

If $p^2M^2 = 2s_1r_2$ these two values coincide, and $s_2 = r_2$, *i.e.* the secondary resistance and reactance are numerically equal.

Equations (14) show that the apparent resistance of the primary circuit is increased, and its apparent reactance is diminished by the presence of the secondary circuit.

From equation (12) we see that the secondary current lags behind the primary P.D. by an angle $\pi - \phi$, where—

$$\tan \phi = \frac{r_1 r_2 - s_1 s_2 + p^2 M^2}{r_1 s_2 + r_2 s_1} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

It is evident from this that the secondary current is in exact opposition to the primary applied P.D., if—

$$r_1 r_2 - s_1 s_2 + p^2 M^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and this condition is satisfied by one value only of $s_2$. Moreover, conditions (16) and (18) cannot be satisfied simultaneously, since then we should have—

$$r_1 s_2 + r_2 s_1 = 0$$

and the secondary current would be infinite, as is seen by reference to equation (12).

If condition (16) is satisfied, we see from equation (11) that the primary current is given by—

$$\dot{i}_1 = \frac{e}{r_1 + \frac{s_1}{s_2} r_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

which shows that, even if the primary current is in phase with the impressed P.D., its value depends upon the resistance of the secondary circuit, and the ratio of the reactances of the two circuits, as well as upon the primary resistance.

Again, from equations (13)—

$$\frac{i_1}{i_2} = \frac{\sqrt{r_2^2 + s_2^2}}{pM}$$

If, now, the secondary resistance is negligible compared with its reactance, and if its reactance is due simply to its self-induction $L_2$, this equation becomes—

$$\frac{\dot{i}_1}{\dot{i}_2} = \frac{L_2}{M}$$

If, further, there is no leakage, and the external circuit is non-reactive, so that (see Ex. 5, p. 9)—

$$M^2 = L_1 L_2$$

$L_1$ being the primary self-induction, we have (see Ex. 9, p. 9)—

$$\frac{\dot{i}_1}{\dot{i}_2} = \sqrt{\frac{L_2}{L_1}}$$

$$= \frac{N_2}{N_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where $N_1$ and $N_2$ are the number of turns on the primary and secondary coils respectively.

Let $V'$ be the P.D. between the secondary terminals.

If $s_2'$, $r_2'$ are the reactance and resistance respectively of the external secondary circuit, we have—

$$V' = r_2' i_2 + k s_2' i_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and the secondary terminal P.D. is in phase with the secondary current if the external circuit is non-reactive. Combining equations (12) and (21), and putting for shortness—

$$r_1 s_2 + r_2 s_1 = a$$
$$r_1 r_2 - s_1 s_2 + p^2 M^2 = b$$

we get—

$$V' = \frac{-(r_2' + k s_2') pMe}{a - kb}$$

$$= \frac{-\{a r_2' - b s_2' + k(b r_2' + a s_2')\} pMe}{a^2 + b^2} \quad . \quad . \quad (22)$$

That is, $V'$ lags behind $e$ by an angle $\pi - \phi$, where—

$$\tan \phi = \frac{b r_2' + a s_2'}{a r_2' - b s_2'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

If the external circuit is non-reactive, this becomes—

$$\tan\phi = \frac{b}{a}$$
$$= \frac{r_1 r_2 - s_1 s_2 + p^2 M^2}{r_1 s_2 + r_2 s_1}$$

If, further, there is no leakage, so that—

$$s_1 s_2 = p^2 M^2$$

it reduces to—

$$\tan\phi = \frac{1}{\frac{s_2}{r_2} + \frac{s_1}{r_1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

which is always a small quantity, but never zero, so that the secondary P.D. lags behind the primary P.D. by nearly 180 degrees in phase.

Now in the general case—

$$\frac{br_2' + as_2'}{ar_2' - bs_2'} = \frac{b}{a} + \frac{1}{a}\,\frac{(a^2 + b^2)s'_2}{ar_2' - bs_2'}$$

is greater than, equal to, or less than—

$$\frac{b}{a}$$

according as—

$$(a^2 + b^2)s_2'$$

is greater than, equal to, or less than 0. That is, the angle of lag $\phi$ increases as the self-induction of the external secondary circuit increases and becomes negative when the reactance becomes negative, due to capacity.

The secondary terminal P.D. is in exact opposition to the primary terminal P.D. when—

$$br_2' + as_2' = 0$$

The output, $W'$, of the transformer is the scalar product of $V'$ and $i_2$. From equations (12) and (22) and § 37, Chap. VII., this can be determined.

Writing (12) in the form—

$$\dot{i}_2 = \frac{-pMe}{a - kb}$$
$$= \frac{-pM(a + kb)e}{a^2 + b^2}$$

we see that the output is given by—

$$W' = \frac{p^2M^2e^2}{(a^2 + b^2)^2}\{a(ar_2' - bs_2') + b(br_2' + as_2')\}$$
$$= \frac{p^2M^2r_2'e^2}{a^2 + b^2}$$
$$= \frac{p^2M^2r_2'e^2}{(r_1s_2 + r_2s_1)^2 + (r_1r_2 - s_1s_2 + p^2M^2)^2} \quad . \quad . \quad (25)$$

Again, the primary current is given by—

$$i_1 = \frac{(r_2 + ks_2)e}{b + ka}$$
$$= \frac{\{br_2 + as_2 + k(bs_2 - ar_2)\}e}{a^2 + b^2}$$

Therefore the input, $W$, being the scalar product of $e$ and $i_1$, is given by—

$$W = \frac{(br_2 + as_2)e^2}{a^2 + b^2}$$
$$= \frac{\{r_2(r_1r_2 - s_1s_2 + p^2M^2) + s_2(r_1s_2 + r_2s_1)\}e^2}{(r_1s_2 + r_2s_1)^2 + (r_1r_2 - s_1s_2 + p^2M^2)^2} \quad . \quad (26)$$

the efficiency $\eta$ is thus given by—

$$\eta = \frac{W'}{W} = \frac{p^2M^2r_2'}{r_1r_2^2 + r_1s_2^2 + r_2p^2M^2}$$
$$= \frac{p^2M^2r_2'}{r_1(r_2^2 + s_2^2) + r_2p^2M^2}$$

which, by the help of (14), may be written in the form—

$$\eta = \frac{(R - r_1)(r_2^2 + s_2^2)r_2'}{r_2R(r_2^2 + s_2^2)}$$
$$= \frac{r_2'}{r_2}\left(1 - \frac{r_1}{R}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Here $r_1$ is the ohmic resistance of the primary coil, $R$ its equivalent resistance, given by equation (14), $r_2$ is the total secondary resistance, and $r_2'$ the resistance of the external secondary circuit.

The above expression for the efficiency does not take into account the iron losses due to hysteresis and eddy currents, but these can be allowed for in the following manner:—

The iron losses are $ei_h$ where $i_h$ is the energy component of the primary current on open secondary circuit. If we write—

$$r_1'i_1^2 = r_1i_1^2 + ei_h \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

the amended expression for the efficiency becomes—

$$\eta' = \frac{r_2'}{r_2}\left(1 - \frac{r_1'}{R}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

since $i_1$ is known for any load and $i_h$ can be calculated from the known iron losses, the value of $r_1'$ is perfectly definite.

**61. To calculate the Energy lost in Eddy Currents in the Core of a Transformer.**—Let Fig. 38 show a section of one of the transformer stampings at right angles to the magnetic flux, so that $b$ is the thickness and $a$ the width of the stamping. Let $L$ be its length.

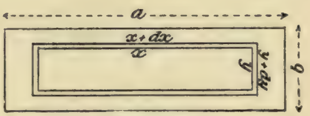


FIG. 38.

Consider an elementary path, whose outside length is $x + dx$ and breadth $y + dy$, and let the induction be given by—

$$B \sin pt$$

The flux through the elementary circuit is—

$$xyB \sin pt$$

therefore the E.M.F. acting round the path is given by—

$$e = \frac{d}{dt}(xyB \sin pt) \times 10^{-8}$$

$$= \frac{xyBp \cos pt}{10^8}$$

The resistance of the elementary path is given by—

$$r = \frac{2x}{\rho L dy} + \frac{2y}{\rho L dx}$$

$\rho$ being the specific resistance of iron—

$$= \frac{2x}{\rho L dy}, \text{ nearly}$$

since $y$ is always small compared with $x$.

The rate at which energy is lost in the elementary path is, therefore, given by—

$$dW = \frac{e^2}{r}$$

$$= \frac{xy^2B^2p^2\rho L}{2 \times 10^{16}} \cos^2 ptdy$$

$$= \frac{ay^3B^2p^2\rho L}{2b \times 10^{16}} \cos^2 ptdy$$

since—

$$a : b = x : y$$

The rate at which energy is lost in the whole stamping is, therefore, given by integrating from $y = o$, to $y = b$, and is—

$$W = \frac{ab^3B^2p^2\rho L}{8 \times 10^{16}} \cos^2 pt$$

$$= \frac{b^2VB^2p^2\rho}{8 \times 10^{16}} \cos^2 pt$$

where $V = abL$ is the volume of the stamping.

The rate at which energy is lost per cubic centimetre is, therefore—

$$\frac{b^2B^2p^2 \cos^2 pt}{8 \times 10^{16}} \text{ watts}$$

and the average rate per second at which energy is lost per cubic centimetre is given by—

$$E = \frac{b^2B^2p^2\rho}{8 \times 10^{16}} \cdot \frac{1}{T}\int_0^T \cos^2 pt \,.\, dt$$

where $T$ is the periodic time—

$$= \frac{b^2B^2p^2\rho}{16 \times 10^{16}}$$

$$= \frac{\pi^2b^2n^2B^2\rho}{4 \times 10^{16}}$$

where $n$ is the frequency of the flux. If $b$ is expressed in mils., this becomes—

$$\tfrac{100}{64}(bnB)^2 \times 10^{-16} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

This value is, however, probably too large, and the value $\left(\frac{bnB}{10^8}\right)^2$ will be more approximate, since the eddy currents will not generally take symmetrical paths as is here assumed, owing to the iron not being quite homogeneous.

Expressing $E$ in ergs per cubic centimetre per cycle we get—

$$E = \frac{(bnB)^2}{10^9}$$

Thus the loss due to eddy currents is seen to be proportional to the square of the thickness of the iron stampings.

Now the counter E.M.F. in the primary circuit is on open secondary equal in volts to the product of the rate of change of the flux and the number of primary turns divided by $10^8$. Thus the counter E.M.F. is proportional to the product of the frequency and the maximum induction; that is, to $nB$. But the counter E.M.F. is nearly equal to the primary applied P.D. on open secondary; therefore we may say approximately that the eddy current loss is proportional also to the square of the applied potential difference, and is constant so long as the applied P.D. remains the same, and is independent of the frequency. Hence, also, if the applied P.D. and the frequency are constant, so is the maximum induction, and the maximum induction for a constant applied P.D. varies inversely as the frequency.

## Hysteresis Loss.

**62.** We have already seen (see § 59) that the average rate of loss of energy due to hysteresis is given by—

$$E' = knB^a \text{ watts}$$

where $a$ is a constant which lies between 1·4 to 1·6, according to the quality of the iron, and $k$ is also a constant for a given quality of iron, and has values given, according to Steinmetz, by the following table:—

TABLE III.

Hysteretic Constants for Different Materials.

| Material. | Hysteretic constant $k$ |
|---|---|
| Very soft iron wire ... ... ... ... ... | 0·002 |
| Very thin soft sheet iron ... ... ... ... | 0·0024 |
| Thin good sheet iron ... ... ... ... | 0·003 |
| Thick sheet iron ... ... .. ... ... | 0·0033 |
| Most ordinary sheet iron for transformer cores ... | 0·004 to 0·0045 |
| Soft annealed cast steel ... ... ... ... | 0·008 |
| Soft machine steel ... ... ... ... ... | 0·0094 |
| Cast steel ... ... ... ... ... ... | 0·012 |
| Cast iron ... ... ... ... ... ... | 0·0162 |
| Hardened cast steel ... ... ... ... | 0 025 |

The values of the hysteresis losses in good soft iron, in microwatts per cubic centimetre for a frequency of 50 alternations per second, for various values of the maximum induction, are given in the following table :—

TABLE IV.

| Maximum induction. $B$. | Value of $B$. | Hysteresis loss in microwatts per cubic centimetre. | | | |
|---|---|---|---|---|---|
| | | = 0·002 | = 0·003 | = 0·004 | = 0·005 |
| 1,000 | 63,100 | 631 | 946 | 1,262 | 1,577 |
| 2,000 | 191,300 | 1,913 | 2,869 | 3,826 | 4,782 |
| 3,000 | 365,900 | 3,659 | 5,483 | 7,381 | 9,147 |
| 4,000 | 850,000 | 5,800 | 8,700 | 11,600 | 14,500 |
| 5,000 | 828,800 | 8,288 | 12,432 | 16,576 | 20,720 |
| 6,000 | 1,111,000 | 11,110 | 16,665 | 22,220 | 27,775 |
| 7,000 | 1,420,000 | 14,200 | 21,300 | 28,400 | 35,500 |
| 8,000 | 1,758,000 | 17,580 | 26,370 | 35,160 | 43,950 |
| 9,000 | 2,122,000 | 21,220 | 31,830 | 42,440 | 53,050 |
| 10,000 | 2,511,000 | 25,110 | 37,665 | 50,220 | 63,775 |

**63. Effect of Shape of P.D. Curve on Iron Losses in Transformer Cores.**—We have seen that, as a rule, we cannot in general expect the wave-form of the P.D. applied between the primary terminals of a transformer to be a true sine curve. It may, in fact, have almost any shape of a periodic nature. It becomes a matter of interest, therefore, to inquire how the iron losses in transformer cores are affected by the shape of the P.D. curve. We proceed to attack the problem from a theoretical point of view.

Assuming that the iron losses are independent of the load on the transformer, we may proceed on the supposition that the secondary circuit is open. Let $v$ be the instantaneous counter E.M.F. in the primary circuit, $A$ the cross-sectional area of the core, $N$ the number of primary turns, $b$ the instantaneous value of the induction, and $S$ the area of the $v$ curve taken over half a period.

We then have, neglecting leakage—

$$v = AN\frac{db}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Integrating this over half a period, we get—

$$S = \int_0^{\frac{T}{2}} vdt$$

$$= 2AN \int_0^B \frac{db}{dt} dt$$

$$= 2AN \int_0^B db$$

$$= 2ANB \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

where $B$ is the maximum value of the induction, and $T$ is the periodic time.

If we assume that the hysteresis loss $W_1$ is given by (Steinmetz's formula)—

$$W_1 = knB^a$$

where $a$ and $k$ are constants, and $n$ is the frequency, we get, by substitution—

$$W_1 = kn\left(\frac{S}{2AN}\right)^a$$

Again, the eddy-current loss is given by (see p. 102)—

$$W_2 = \frac{m}{T} \int_0^T \left(\frac{db}{dt}\right)^2 dt$$

where $m$ is a constant, by (31)—

$$= \frac{m}{T} \int_0^T \left(\frac{v}{AN}\right)^2 db \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

$T$ being the periodic time. Therefore—

$$W_2 = \frac{m}{A^2N^2} \times (\text{mean square value of primary counter E.M.F.})$$

The total iron losses are therefore given by—

$$W = W_1 + W_2$$

$$= kn\left(\frac{S}{2AN}\right)^a + \frac{m}{A^2N^2} \times (\text{mean sq. value of primary counter E.M.F.})$$

Now, on open secondary circuit, the primary P.D. is almost

exactly equal to the primary counter E.M.F., but opposite in phase; we may therefore write—

$$W = kn\left(\frac{S}{2AN}\right) + \frac{m}{A^2N^2} \times (\text{mean square P.D.}) \quad . \ (34)$$

If, then, the P.D. between the primary terminals, as measured by a hot wire or electrostatic voltmeter, is kept constant, the total iron loss is given by equation (34), whatever be the **shape** of the wave.

The problem of finding the area of the wave which makes the iron losses a minimum is the same as that of making the area of the P.D. wave as small as possible while the R.M.S. potential difference, $V$, remains constant. We have, therefore, to make—

$$\int V dt = \text{a minimum}$$

while—

$$\int V^2 dt = \text{a constant} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

From this we deduce that $\int V dt$ has no absolute minimum, but may be made as small as we please by making the maximum value of $V$ correspondingly large and the P.D. curve narrow and peaky.

It is to be noticed that the hysteresis losses only are affected by the wave form of the primary P.D. as the eddy current losses are constant when the P.D. is constant.

It is not advisable to attempt to reduce the hysteresis loss too much in the manner here indicated, as, by so doing, the maximum P.D. becomes very great, and the insulation of the transformer must be correspondingly increased, a course which would produce a very costly appliance.

These theoretical results have been amply proved by experimental investigation, and we would draw the attention of our readers to the writings of Mr. Steinmetz,[1] Dr. Fleming,[2] Messrs. Beeton, Taylor, and Barr,[3] Dr. Roessler,[4] Mr. Evershed,[5] and Mr. Feldman,[6] on the subject.

[1] *Electrician*, August 24, 1894, vol. xxxiii. p. 498.

[2] *Ibid.*, June 28, 1895, vol. xxxv. p. 304; January 10, 1896, vol. xxxvi.

[3] *Ibid.*, June 21, 1895, vol. xxxv. p. 257; June 28, 1895, vol. xxxv. p. 286; November 8, 1895, vol. xxxvi. p. 61.

[4] *Ibid.*, November 22, 1895, vol. xxxvi. p. 124; December 6, 1895, vol. xxxvi. p. 222.

[5] *Ibid.*, March 27, 1891, vol. xxvi. p. 635.

[6] *Ibid.*, October 18, 1895, vol. xxxv. p. 809.

Perhaps the experimental investigation of greatest interest is that of Messrs. Beeton, Taylor, and Barr, on account of an ingenious method of obtaining various wave forms from a single alternator by the use of a piece of apparatus called an **injector,** by means of which resistance, self-induction, or capacity can be thrown in the circuit at any instant during the period at will, producing wave forms of the P.D. which could only have been obtained otherwise by a series of dissimilar alternators.

The conclusions arrived at experimentally are—

1. That if the R.M.S. value of the applied P.D. is constant, and the area of the P.D. wave is constant, then, whatever be the shape of this wave, the total iron loss cannot vary.

2. That if the R.M.S. value of the applied P.D. is constant, but the area of the P.D. wave varies, then, whatever be the shape of the wave, the total iron loss will vary by an amount which is only dependent upon the area of the P.D. wave.

These results are in complete agreement with the preceding theory.

## Tests of Transformers.

**64.** Tests on Ferranti standard transformers furnish the following results:—

TABLE V.

Frequency 50 Cycles per Second.

Transformation Ratio.

| Output in kilowatts. | Efficiencies. | | | | | Loss at no-load per cent. | Secondary drop on full non-inductive load per cent. | Secondary load on full inductive load. Power factor 0·8 per cent. |
|---|---|---|---|---|---|---|---|---|
| | Full load. | $\frac{3}{4}$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{10}$ | | | |
| 10 | 96·4 | 96·5 | 96·3 | 94.2 | 87·6 | 1·4 | 2 | 4·5 |
| 20 | 97·0 | 97·2 | 97·0 | 95·2 | 89·6 | 1·15 | 2 | 4·5 |
| 30 | 97·5 | 97·6 | 97·4 | 95·9 | 91·0 | 1·0 | 2 | 4·5 |
| 40 | 97·6 | 97·7 | 97·5 | 96·1 | 91 5 | 0·93 | 2 | 4·5 |
| 50 | 97·7 | 97·8 | 97·7 | 96·3 | 91·9 | 0 88 | 2 | 4·5 |

TABLE VI.

FREQUENCY 100 CYCLES PER SECOND.

Transformation Ratio.

| Output in kilowatts. | Efficiencies. | | | | | Loss at no-load per cent. | Secondary drop on full non-inductive load per cent. | Secondary load on full inductive load. Power factor 0·8 per cent. |
|---|---|---|---|---|---|---|---|---|
| | Full load. | $\frac{3}{4}$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{10}$ | | | |
| 10 | 96·9 | 97·0 | 96·7 | 94·6 | 88·4 | 1·3 | 2 | 6 |
| 20 | 97·4 | 97·6 | 97·3 | 95·9 | 90·9 | 1·0 | 2 | 6 |
| 30 | 97·7 | 97·8 | 97·6 | 97·3 | 91·8 | 0·9 | 2 | 6 |
| 40 | 97·8 | 97·9 | 97·8 | 96·5 | 92·1 | 0·85 | 2 | 6 |
| 50 | 97·9 | 98·1 | 97·9 | 96·6 | 92·5 | 0·8 | 2 | 6 |

## PROBLEMS ON CHAPTER XI.

1. Show that, if the primary and secondary coils of a transformer are so wound that there is no leakage, the equivalent self-induction of the primary circuit with closed secondary is given by—

$$\frac{L_1 r_2^2}{r_2^2 + p^2 L^2}$$

where $L_1$ and $L_2$ are the self-inductions of the primary and secondary coils respectively, $r_2$ is the total resistance of the secondary circuit, and $p = 2\pi n$, $n$ being the frequency, the secondary external circuit being non-inductive.

2. Assuming the result given in Question 1, show that in a transformer having no leakage, and non-inductive external secondary circuit, the primary current lags behind the applied by P.D. an angle $\phi$ given by—

$$\tan \phi = \frac{pL_1 r_2^2}{r_1 r_2 + p^2 L_2 (r_1 L_2 + r_2 L_1)}$$

where $r_1$ is the resistance of the primary coil.

3. Show that, in a transformer having no magnetic leakage, the impedance of the primary on closed non-inductive secondary circuit is greater than, equal to, or less than its impedance on open secondary, according as $\sqrt{2r_1 r_2}$ is greater than, equal to, or less than $pM$, where $r_1$ is the resistance of the primary coil, $r_2$ the total resistance of the secondary circuit, $M$ the mutual induction of the primary and secondary coils, and $p = 2\pi n$, $n$ being the frequency.

4. Show that, taking leakage into account, the condition in Question 3 is obtained by writing—

$$\sqrt{M^2 - L_1 L_2} \text{ for } M$$

where $L_1$ is the self-induction of the primary coil, and $L_2$ that of the secondary coil.

5. A condenser of capacity $C$ is permanently connected in parallel with the primary coil of a transformer: show that the power factor will be unity, if—

$$C = \frac{L}{R^2 + p^2 L^2}$$

where—

$$L = L_1 - \frac{p^2 L_2 M^2}{r_2^2 + p^2 L_2^2}$$

$$R = r_1 + \frac{p^2 M^2 r_2}{r_2^2 + p^2 L_2^2}$$

the notation being as in Question 3.

6. In a transformer having no magnetic leakage the self-inductions of the primary and secondary coils are respectively 1 and 0·1 henry, the resistances 10 ohms and 0·1 ohm, the mutual induction of the two coils 0·09, and the frequency 100 periods per second: what capacity must be placed in parallel with the primary coil to give unit power factor?

*Answer.* About 2·8 microfarads.

7. A transformer has a primary and two secondary coils; the resistance and self-induction of the primary are 10 ohms and 0·001 henry respectively; those of the first secondary are 5 ohms and 0·0005 henry respectively, and those of the second secondary are 1 ohm and 0·0001 henry respectively; the mutual inductions between the primary and first secondary, the primary and second secondaries, and the two secondaries, are respectively 0·00065, 0·0003, and 0 0002 henry. If the primary P.D. is 1000 volts, what is the primary current, and what is the difference of phase between P.D. and current, the frequency being 100?

*Answer.* 98·52 amperes; 6° 10′.

8. The P.D. between the primary terminals of a transformer is 2000 volts, the current is 0·15 ampere, and the power as measured by a wattmeter is 275 watts: what is the power factor?

*Answer.* 91·67 per cent.

9. Show that, if the resistance of the secondary coil of a transformer is negligible, its efficiency is a maximum when the external secondary resistance is numerically equal to the total reactance of the secondary circuit.

# CHAPTER XII.

Design of Transformers—Effects of Magnetic Leakage—Core Laminations—Methods of testing Transformers—Calculation of Hysteresis Loss—Regulation of Transformers—Insulation and Temperature Tests.

## DESIGN OF TRANSFORMERS.

**65.** We do not propose to enter into an exhaustive treatment of the design of transformers, as we should be compelled to study closely several types, the general principles of all of which are the same, and the differences only in details of construction.

We shall, therefore, only indicate the nature of the calculations involved, referring at the same time to such practical points as depend upon previous experience.

**66.** Before commencing a design, we must have the following data:—

(1) The full-load output ($W$ watts) of the transformer.

(2) The primary impressed P.D., $E_1$ volts.

(3) The secondary terminal P.D., $E_2$ volts.

(4) The copper and iron losses in percentages of the full-load output.

(5) The maximum induction density, $B$, at which the iron of the transformer is to be worked.

(6) The frequency, $n$, of the primary supply current.

(7) The thickness and permeability of the core stampings.

(8) The type of transformer to be made.

(9) The maximum permissible rise of temperature.

We propose to give approximate calculations in the following particular case.

Let $W = 10{,}000$ watts.
$E_1 = 1{,}000$ volts.
$E_2 = 100$ volts.
$B = 2{,}500$ lines per square centimetre.
$n = 50$ complete cycles per second.

Let the iron and copper losses be each 150 watts at full load. Working at the same current density in both the primary and secondary circuits, the consequent dissipation of energy in each will be 75 watts.

**67. Copper Circuits.**—Let $l_1$ be the length of the primary circuit, $\sigma_1$ its cross-section, and $r_1$ its resistance; and let $l_2$, $\sigma_2$, $r_2$ have similar meanings for the secondary circuit. If $\rho$ is the specific resistance of copper, we have—

$$\left.\begin{aligned} r_1 &= \frac{\rho l_1}{\sigma_1} \\ r_2 &= \frac{\rho l_2}{\sigma_2} \end{aligned}\right\} \quad \ldots \ldots \quad (1)$$

The full-load primary current is—

$$\begin{aligned} i_1 &= \frac{W}{E_1} \\ &= 10 \text{ amperes (approximately)} \quad \ldots \quad (2) \end{aligned}$$

and the secondary current is—

$$i_2 = 100 \text{ amperes} \quad \ldots \ldots \quad (3)$$

We have, therefore—

$$i_1^2 r_1 + i_2^2 r_2 = 150 \text{ watts}$$

If the two circuits are wound with wire, so that the current density in them is the same, we have—

$$i_1^2 r_1 = i_2^2 r_2 = 75 \text{ watts} \quad \ldots \ldots \quad (4)$$

and—

$$\frac{i_1}{\sigma_1} = \frac{i_2}{\sigma_2} = 160 \text{ amperes per square centimetre}$$

working at about 1000 amperes per square inch, therefore—

$$\begin{aligned} \sigma_1 &= \frac{i_1}{160} \\ &= \tfrac{10}{160} \\ &= 0{\cdot}0625 \text{ square centimetre} \quad \ldots \quad (5) \end{aligned}$$

and—

$$\begin{aligned} \sigma_2 &= \frac{i_2}{160} = \tfrac{100}{160} \\ &= 0{\cdot}625 \text{ square centimetre} \quad \ldots \quad (6) \end{aligned}$$

Again, from equation (4) we get the resistances of the two circuits to be—

$$r_1 = \frac{75}{10^2}$$
$$= 0{\cdot}75 \text{ ohm, nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$
$$r_2 = \frac{75}{100^2}$$
$$= 0{\cdot}0075 \text{ ohm, nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Also taking $\rho = 16 \times 10^{-7}$, we have, by (1)—

$$l_1 = \frac{\sigma_1 r_1}{\rho}$$

that is—

$$= \frac{0{\cdot}0625 \times 0{\cdot}75 \times 10^7}{16}$$
$$= 29297 \text{ centimetres, nearly} \quad . \quad . \quad . \quad (9)$$

and

$$l_2 = \frac{\sigma_2 r_2}{\rho}$$
$$= \frac{0{\cdot}625 \times 0{\cdot}0075 \times 10^7}{16}$$
$$= 0{\cdot}2930 \text{ centimetre, nearly} \quad . \quad . \quad . \quad (10)$$

We shall have to leave the determination of the number of primary, $N_1$, and secondary, $N_2$, turns until the dimensions of the iron core have been found.

Summarizing the items already determined, we have—

$i_1 = 10$ amperes
$i_2 = 100$ amperes
$\sigma_1 = 0{\cdot}0625$ square centimetre
$\sigma_2 = 0{\cdot}625$ square centimetre
$l_1 = 29297$ centimetres
$l_2 = 2930$ centimetres
$r_1 = 0{\cdot}75$ ohm
$r_2 = 0{\cdot}0075$ ohm.

**68. Iron Circuit.**—The dimensions of the iron circuit are determined from the assumed iron losses.

Let $A$ be the cross-sectional area of the core in square centimetres, and $L$ the mean length of the magnetic path, so that—

$AL$ = approximate volume of iron core in cubic centimetres

The iron losses are due to hysteresis and eddy currents. If the core is built up of soft annealed iron stampings, the hysteresis loss can be found from Table II., Chap. X., and § 62. The maximum induction was taken to be 2500 lines per square centimetre; hence the ergs lost per cycle per cubic centimetre are 600.

With a frequency of 50 complete cycles per second, this gives the loss in watts per cubic centimetre to be—

$$h = \frac{600 \times 50}{10^7}$$
$$= 0{\cdot}003 \text{ watts} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The watts lost per cubic centimetre due to eddy currents are calculated from the formula (see Chap. XI. § 61)—

$$h' = b^2 n^2 B^2 \times 10^{-16}$$

where $b$ is the thickness of the stampings in mils.

Suppose that $b$ = 15 mils., then in the present case—

$$h' = 15^2 \times 50^2 \times 2500^2 \times 10^{-16}$$
$$= 0{\cdot}00035 \text{ watts} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The total iron loss in watts per cubic centimetre is, therefore, given by—

$$H = h + h'$$
$$= 0{\cdot}00335 \text{ watts} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Since the total iron loss is 150 watts, the volume of the core is given by—

$$V = \frac{150}{0{\cdot}00335}$$
$$= 44776 \text{ cubic centimetres}$$

that is—

$$AL = 44776 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where $A$ is the cross-sectional area of the core, and $L$ is the mean length of the magnetic path.

As the insulation between the stampings will occupy 10 or 15

per cent. of the whole volume, $V$ should be correspondingly increased to about 51493 cubic centimetres.

Now—

$$E_1 \doteq \frac{AN_1}{\sqrt{2}} \cdot \frac{1}{10^8} \times \text{maximum value of } \frac{db}{dt}$$
$$= \frac{AN_1 \times 2\pi nB}{10^8\sqrt{2}}$$

therefore—

$$AN_1 = \frac{10^8 E_1}{\sqrt{2}\pi nB}$$
$$= \frac{10^8 \times 1000}{\sqrt{2} \times \pi \times 50 \times 2500}$$
$$= 180000, \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Suppose, now, that the primary circuit is wound with 230 turns and the secondary circuit with approximately 23 turns. The primary coil will be wound with the equivalent of No. 10 S.W.G. wire, and will have an area, when double cotton-covered, of about 0·0177 square inch. The secondary wire will be equivalent to No. $3/_0$ S.W.G., and will, including insulation, have an area of about 0·126 square inch.

The primary coil will, therefore, require a winding space of about 5·175 square inches, and the secondary coil a winding space of about 3·68 square inch.

Thus the total winding space required for the two coils is 9 square inches, or, allowing 25 per cent. for insulation between the two coils and between different layers of the primary, we may take the winding space required to be 11·25 square inches, or 73 square centimetres.

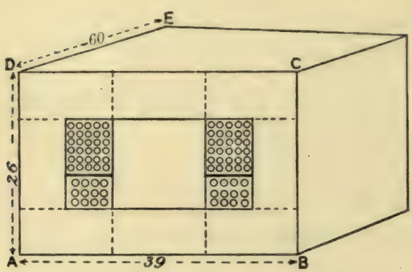


FIG. 39.

Suppose, now, that we select a transformer of the Mordey type (see Fig. 39) in which the width, $DC$, bears to the depth, $CB$, the ratio of 3 : 2. The length, $DE$, may be altered to suit the circumstances of the case.

If $DC = 6x$, and $DA = 4x$, then the winding space is (see Fig. 39)—

$$4 \cdot x^2 = 73$$

whence $x = 4 \cdot 27$ centimetres. This would give too small a cooling surface, so we take $x = 6 \cdot 5$.

Now, from equation (15) we get—

$$A = \frac{180000}{230}$$
$$= 281 \cdot 4 \text{ square centimetres} \quad . \quad . \quad . \quad . \quad (16)$$

therefore the length $DE$ of the transformer is—

$$\frac{783}{13} \text{ centimetres}$$

or—

$$60 \text{ centimetres}$$

Again, the resistance of the secondary coil is 0·0075 ohm; therefore the drop of volts due to secondary resistance from no-load to full-load is 0·75 volt; an extra turn on the secondary will more than compensate for this.

We have hitherto assumed that the primary applied P.D. and the primary counter E.M.F. are the same, because it is the ratio of the primary counter E.M.F. to the secondary induced E.M.F., that is the same as the ratio of the turns. The drop of volts in the primary coil itself is—

$$0 \cdot 75 \times 10 = 7 \cdot 5 \text{ volts}$$

that is, 0·75 per cent., so that, neglecting magnetic leakage, the secondary induced E.M.F. is given by—

$$\frac{40}{400} \times 992 \cdot 5 = 99 \cdot 25 \text{ volts}$$

Taking into account the drop of volts in the secondary coil itself, the secondary terminal P.D. is, neglecting leakage, 98·5 volts; and allowing 2 per cent. for magnetic leakage between the primary and secondary coils, the secondary terminal P.D. becomes 96·5 volts. If we put 24 turns on the secondary, the terminal P.D. becomes 103.

**69.** The next step is to calculate the primary current on open secondary circuit. This is—

$$i_1 = i_\mu + i_h$$

where $i_\mu$ is the magnetizing wattless component, and $i_h$ is the hysteretic energy component.

The component $i_\mu$ is determined as follows. We have—

$$B = \mu H$$

where $\mu$ is the permeability of the iron core, and $H$ is the magnetizing force. Also—

$$\frac{H}{\sqrt{2}} = \frac{4\pi}{10} \cdot \frac{N_1 i_\mu}{L}$$

where $i_\mu$ is the R.M.S. value of the wattless component, and $H$ the maximum value of the magnetizing force. Therefore—

$$i_\mu = \frac{BL}{1{\cdot}76 \mu N_1}$$

Substituting the values $B = 2500$, $L = 65$, $N = 230$, and $\mu = 2390$, we get—

$$i_\mu = \frac{2500 \times 65}{1{\cdot}76 \times 2390 \times 400}$$
$$= 0{\cdot}167 \text{ ampere} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Also, $i_h$ is calculated from the fact that the primary impressed P.D. multiplied by $i_h$ is equal to the total iron loss, so that—

$$1000 i_h = 150$$
$$i_h = 0{\cdot}15 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Now, $i_\mu$ and $i_h$ are in quadrature; therefore—

$$i = \sqrt{0{\cdot}15^2 + 0{\cdot}167^2}$$
$$= 0{\cdot}225 \text{ ampere} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The power factor on open secondary circuit is given by—

$$\cos\theta = \frac{i_h}{i}$$
$$= \frac{0{\cdot}15}{0{\cdot}225}$$
$$= 0{\cdot}67 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

To find the watts lost by radiation and conduction of heat per square inch of cooling surface at no-load and full-load respectively.

The surface of the transformer is—

$$= 9700 \text{ square centimetres}$$
$$= 1497 \text{ square inches}$$

At no-load the number of watts lost per square inch of surface is—

$$\tfrac{150}{1497} = 0{\cdot}1$$

and at full-load—

$$\tfrac{300}{1497} = 0{\cdot}2$$

which would be sufficiently small.

Actual tests show that a cooling surface of 2000 square centimetres allows a rise of temperature, under normal conditions, of 1° C. per watt dissipated in the transformers. The rise of temperature in the present case will, therefore, not exceed—

$$300 \times \tfrac{2000}{9700}$$
$$= 62^\circ \text{ C.}$$

This, in fact, will be somewhat in excess of the truth, since, owing to the shape of the windings, the cooling surface will be greater than 9700 square centimetres.

The total volume of iron in the transformer is, by reference to Fig. 39, seen to be—

$$= 50700 \text{ cubic centimetres} \quad . \quad . \quad . \quad . \quad (21)$$

This gives a loss of only 0·003 watts per cubic centimetre, which is a liberal allowance of iron.

To find the weight per kilowatt of the transformer.

Since iron weighs 0·017 lb. per cubic centimetre, the weight of iron is about 862 lbs.

Also the weight of copper is 80 lbs.

Thus the weight of the transformer per kilowatt of output is—

$$\tfrac{942}{10} = 94{\cdot}2 \text{ lbs. per kilowatt}$$

### Effects of Magnetic Leakage.

**70.** If there is no magnetic leakage, the square of the coefficient of mutual induction of the primary and secondary coils is

equal to the product of their respective coefficients of self-induction. That is—

$$M^2 = L_1 L_2$$

or—

$$M = \sqrt{L_1 L_2}$$

As, however, there is always some leakage, we may write—

$$M = (1 - \lambda)\sqrt{L_1 L_2}$$

where $\lambda$ is less than unity, and is a measure of the leakage.

Leakage causes, then, a diminution of the mutual induction, and, consequently, a drop in the induced secondary E.M.F. At the same time, it causes an increase in the equivalent self-induction of the primary circuit (see § 60, Chap. XI.). Now, it is the equivalent self-induction of the primary circuit which causes the primary current to lag behind the applied P.D., and, consequently, produces a larger primary current than corresponds to the secondary current in the ratio of their respective turns.

The principal effects of magnetic leakage are, therefore—

(1) The production of an excessive wattless primary current.

(2) A consequent increase in the primary copper loss.

(3) An increased drop of the secondary terminal P.D.

(4) A diminution in the efficiency of the transformer, owing to the increased primary copper loss.

## Core Laminations.

**71.** We have seen (see § 61, Chap. XI.) that the loss in the core of a transformer, due to eddy currents, is proportional to the square of the thickness of the core stampings. This would make it appear that the thinner the stampings the better would be the results obtained.

So far as eddy-current losses alone are concerned, this is correct; but there are other considerations which impose a minimum thickness of the plates. Suppose that with stampings 18 mils. thick the insulation between the stampings occupied 15 per cent. of the whole volume of the core; then, with the same kind of insulation and stampings 9 mils. thick, the insulation would occupy about 26·1 per cent. of the volume of the core, and so, by making the stampings thinner and thinner, the ratio of the volume

of insulation to that of iron becomes greater and greater, and for a given volume, the magnetic flux through the core less and less.

On the other hand, there is an upper limit to the thickness of the stampings, for when iron is subjected to an alternating magnetizing force, the induced currents only penetrate to a depth depending upon the permeability, the frequency, and the specific resistance. This is known as the "skin" effect, and it can be shown mathematically[1] that we may take as a measure of the thickness of the "skin," the expression—

$$\sqrt{\frac{\sigma}{2\pi\mu p}}$$

where $\sigma$ is the specific resistance of the iron, $\mu$ its permeability, and $p$ is $2\pi$ times the frequency.

If $\mu = 1000$, and the frequency is 100 periods per second, the skin for soft iron is about half a millimetre thick, and for a frequency of 50 periods the skin is about 0·7 millimetre, or about 27·3 mils. thick. Thus, unless the laminations are less than 27·3 mils., they produce no beneficial effect.

## Methods of testing Transformers.

**72.** There are several methods of testing transformers for efficiency, each of which has its advantages in special cases. In each method the watts given to the primary circuit and taken from the secondary circuit at various loads are measured or calculated, and the efficiency determined as the ratio of the output to the input.

**73. First Method (Wattmeter).**—Suppose that we have at our disposal the transformer whose efficiency is to be determined, an alternator, or some source, capable of supplying power to the transformer from no-load to full-load, an alternating-current ammeter and voltmeter, a wattmeter, and a set of non-inductive resistances, with switches and connecting cable.

The terminals of the alternator should be connected through a resistance to the primary terminals of the transformer. The object of this resistance is merely to prevent an abnormal rush of

[1] See "Recent Researches in Electricity and Magnetism," J. J. Thomson, Clarendon Press. First edition, 1893, p. 281.

current at the moment of switching the alternator on to the transformer, which may occur if the residual magnetism in the core happens to be in the same direction as the magnetizing force due to the first current wave. In this case the E.M.F. of self-induction of the primary would at first be comparatively small, and unless the precaution is taken of inserting a starting resistance the first rush of current, although not likely to injure the transformer, may be large enough to blow any fuses in the circuit or damage the current-measuring instruments. Before taking any readings, the starting resistance is, of course, cut out.

In the primary circuit must be placed a wattmeter, $W$ (see Fig. 40), the fine-wire coil of which is connected across the primary terminals of the transformer, $T$, while the thick coil is placed in series with the primary of $T$, and carries the main current from the alternator $D$. The secondary circuit of the transformer is closed through an ammeter, $A$, and a non-inductive variable resistance, $R$, and a voltmeter, $V$, is placed across the secondary terminals. $S$ is the starting resistance in the primary circuit.

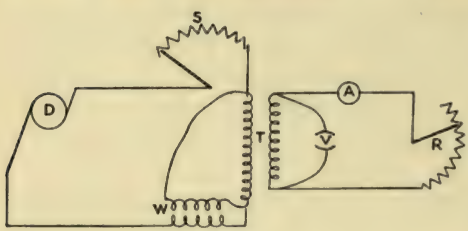

FIG. 40.

The primary current will not, in general, coincide in phase with the primary $PD$; hence the necessity of using a wattmeter to measure the input.

The secondary current will coincide in phase with the secondary terminal $PD$, since the load is non-inductive, so that the product of the ammeter and voltmeter readings gives the output in watts.

Let the reading of the wattmeter when the transformer is at the given load $= w$ watts, and let the readings of the ammeter and voltmeter be $i$ amperes and $v$ volts respectively.

The efficiency, $\eta$, is then given by—

$$\eta = \frac{iv}{w}$$

By altering the resistance in the secondary external circuit the efficiency at any load may be determined.

The ammeter and voltmeter in the secondary circuit might, of course, be replaced by a wattmeter if desired; and any system of instruments which would give the true input of the transformer might replace the wattmeter in the primary circuit.

**74. Second Method.**—In the first method it is assumed that an alternator is at hand which is capable of supplying power to the transformer when working on full-load. If the transformer is a large one, and if the alternators at disposal are of insufficient capacity to supply the transformer at full-load, other methods of determining the efficiency must be adopted.

The output of the transformer at any load is the difference between the input and the losses. These losses consist of (1) hysteresis loss, (2) eddy-current loss, and (3) copper ($i^2r$) losses in both primary and secondary circuits. The iron losses are practically the input of the transformer on open secondary circuit; and the copper losses can be calculated for any assumed load, and hence the efficiency corresponding to that load may be determined.

Let $W$ be the iron losses.
$r_1$ the primary resistance.
$r_2$ the secondary internal resistance.
$i_1$ the primary current corresponding to the assumed output.
$i_2$ the corresponding secondary output.
$w$ the assumed output in watts.
$\eta$ the efficiency.

Then—

$$\eta = \frac{\text{output}}{\text{output} + \text{losses}}$$

$$= \frac{w}{w + W + i_1{}^2 r_1 + i_2{}^2 r_2}$$

The copper losses on open secondary circuit are so small that they may be neglected. We may, therefore, determine $W$ by placing a wattmeter in the primary circuit when the secondary circuit is open.

The next step is to measure, by means of a Wheatstone's bridge (or otherwise), the resistances $r_1$ and $r_2$ of the two coils, correcting for the temperature corresponding to the load.

By giving the assumed values to $w$, the efficiency at any output may be calculated.

**75. Third Method.**—This method, due to Mr. Mordey, consists in running the transformer at the load at which it is to be tested until its temperature, as determined by a thermometer placed inside it, is constant. The alternating current is now replaced by a steady current, having such a value as to maintain the same steady temperature. The total losses are then given by the product of the steady current in amperes and the volts between the terminals of the transformer.

**76. Fourth Method.**—This is a method by which the efficiency can be determined by calculation from a knowledge of the primary R.M.S., applied P.D., the resistances of the primary and secondary windings, the frequency, the dimensions of the core, the thickness of the core stampings, and the number of turns on the primary coil.

The resistances of the coils may be calculated by determining the lengths and cross-sections of the wire composing the two coils respectively.

## Calculation of Hysteresis Loss.

**77.** To calculate the hysteresis loss, we must first determine the maximum induction in the core, and then refer to Table IV., Chap. XI., giving the loss in watts per cubic centimetre at that induction.

Let $e$ be the R.M.S. applied primary P.D.
$A$ the cross-section of the core.
$N_1$ the number of primary turns.
$B$ the maximum induction.
$n$ the frequency.

Then—

$$e\sqrt{2} = \frac{2\pi n A N_1 B}{10^8} \text{ very nearly}$$

since this gives the counter E.M.F. in the primary circuit; therefore—

$$B = \frac{10^8 e}{\sqrt{2}\pi n N_1 A}$$

Substituting the known values of $e$, $n$, $N$, and $A$, we obtain the value of $B$. Referring now to Table IV., Chap. XI., we find the

corresponding hysteresis loss in watts per cubic centimetre, and multiplying this by the volume of the core in cubic centimetres, the total hysteresis is determined.

The eddy currents can be calculated from the formula—

$$h' = \left(\frac{bnB}{10^8}\right)^2$$

(see § 61, Chap. XI.), and the copper losses can be calculated for any assumed output on calculating the resistances of the two circuits. We then have—

$$\eta = \frac{\text{assumed output}}{\text{assumed output} + \text{total losses}}$$

We note that the iron losses per cubic centimetre may be written (see §§ 59 and 61)—

$$H = knB^{1 \cdot 6} + \left(\frac{bnB}{10^8}\right)^2$$

$k$ being a constant.

Now, if the applied primary P.D. is constant, the product $nB$ is constant; therefore we have—

$$H = \frac{k(nB)^{1 \cdot 6}}{n^{0 \cdot 6}} + \frac{b^2(nB)^2}{10^{16}}$$
$$= \frac{P}{n^{0 \cdot 6}} + Q$$

where $P$ and $Q$ are constants, if $e$ is constant.

Thus we see that the iron losses diminish as the frequency increases.

## Regulation of Transformers.

**78.** One of the most essential features of a good transformer is that the secondary terminal P.D. should remain constant, as the secondary load varies.

The causes which tend to bad regulation are (1) large primary magnetizing current, (2) magnetic leakage, and (3) large secondary internal resistance.

The first and second are minimized by working at low induction densities, and by having closed magnetic circuits. In modern transformers there is very little leakage, owing to the fact that there is

less tendency to leakage when the magnetic circuit is closed; also by winding the primary and secondary circuits in sections placed alternately side by side the difficulty is to a great extent overcome.

To test the regulation of a transformer the terminal secondary voltage should be taken at all loads from no-load to full-load, the primary terminal P.D. being kept constant.

### INSULATION TESTS.

**79.** Having tested the efficiency and regulation of a transformer, the next step is to test its insulation.

The primary coil should be well insulated from the secondary. To test the quality of the insulation between the two coils requires more than the ordinary measurements for the determination of insulation resistance. The insulation should first be determined by any of the known methods, and then the transformer should be run continuously for two or three hours, with twice or three times the voltage for which it is intended, and then insulation resistance should be determined again. If the insulation has in no way suffered, the test may be considered satisfactory.

### TEMPERATURE TEST.

**80.** It is of great importance that a transformer should remain sufficiently cool even if working continuously on full-load. If the temperature exceeds about 60° C., there is danger of the insulation being damaged.

Before a transformer is passed as satisfactory it should, therefore, be run continuously at full-load for some hours, and temperature readings taken at regular intervals of time by means of thermometers inserted into it at different places. When the readings indicate that the temperature is stationary, the thermometers should not register more than 60° C.

The stationary temperature depends upon the size and construction of the transformer. A small transformer has a greater cooling surface per watt of output than a large one. There is, as a rule, no danger of overheating small transformers by simply running continuously at full-load. With large transformers, however, special devices have frequently to be employed to get rid of the heat, *e.g.* immersion in oil.

## PROBLEMS ON CHAPTER XII.

1. Calculate the lengths, cross-sections, and resistances of the primary and secondary windings of a transformer whose output is 50 kilowatts, the transformation ratio being 2000 to 100 volts, the copper losses 1·25 per cent. of the output, and the current density being 150 amperes per square centimetre.

*Answers.* $l_1 = 52{,}188$ cms.; $\sigma_1 = 0·167$ sq. cm.; $r_1 = 0·5$ ohm
$l_2 = 26{,}039$ cms.; $\sigma_2 = 3·333$ sq. cms.; $r_2 = 0·00125$ ohm

2. If the frequency is 50 cycles per second, and the induction 3500 lines per square centimetre, and the rest being the same as in Question 1, complete the design as a transformer of the Mordey type, and calculate the full-load efficiency.

3. Calculate the lengths, cross-sections, and resistances of the primary and secondary windings of a transformer whose output is 10 kilowatts, the transformation ratio being 500 to 200 volts, the copper loss 4 per cent. of the output, and the current density being 150 amperes per square centimetre.

*Answer.* $l_1 = 41{,}667$ cms.; $\sigma_1 = 0·133$ sq. cm.; $r_1 = 0·5$ ohm
$l_2 = 16{,}667$ cms.; $\sigma_2 = 0·333$ sq. cm.; $r_2 = 0·08$ ohm

4. If the frequency is 80 cycles per second, and the induction 4000 lines per square centimetre, and the rest the same as in Question 3, what is the total weight of the transformer, the iron losses being 5 per cent. of the output, and the thickness of the stampings 15 mils.?

*Answer.* 913 lbs.

## CHAPTER XIII.

Synchronous Motors—Method of synchronizing a Power Plant consisting of Generator and Motor—Armature Reaction—Stability of Plant.

### SYNCHRONOUS MOTORS.

**81.** Just as a direct-current generator can be driven as a motor by applying a continuous P.D. between its terminals, so can an alternator, if its field is separately excited, and a suitable alternating P.D. be applied between its armature terminals.

Suppose that the field of an alternator is excited by a direct current in such a way that its poles are alternately north and south, and suppose an alternating current is sent through its armature coils so that each armature pole is for one half-period of the current a north pole, and for the next half-period a south pole. It is easily seen that there will be impulses on the armature tending to drive it first in one direction and then in the other, so that if the armature is initially at rest, no motion at all will take place. But suppose that a motion is mechanically given to the armature, so that in half a period of the alternating current an armature pole passes from one field pole to the next, then the impulses due to the reaction between field and armature will always tend to drive the armature in the same direction, and motion will continue so long as no external influence causes the armature to revolve at a different rate.

An alternator driven as a motor can therefore only run at one **fixed speed,** depending upon the periodicity of the current which drives it. On this account, such a machine is called a **Synchronous Motor,** since it will only run at that particular speed which corresponds to synchronism with the alternating current which drives it.

On starting such a motor, the armature current is not switched on until the speed corresponding to synchronism is attained by some other means.

**82.** Any piece of apparatus which is used to indicate when synchronism is attained is called a **Synchronizer.**

Let $M$ (Fig. 36) be the motor, and $D$ the distant alternator; let $V$ be a hot-wire or electrostatic voltmeter, and, when working at high voltages, let $R$ be a high resistance; and let the connections be as represented in the diagram before the alternating current is sent round the motor armature. When the motor and generator are not in synchronism, the P.D. between the terminals of the voltmeter will oscillate with a rapidity equal to the difference of the frequencies of the E.M.F.s of generator and motor; and the voltmeter needle will oscillate accordingly. The oscillations will become slower and slower as the speeds approach synchronism; and when synchronism is exactly attained, the needle will be steady, and will have a minimum deflection. Incandescent lamps, or a gold-leaf electroscope, make good synchronizers, and might replace the voltmeter. The above method of obtaining synchronism is obviously applicable to the general case of any two alternating-current machines, whether both generators, or one a generator and the other a motor.

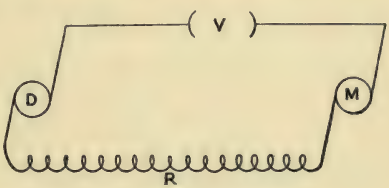

FIG. 41.

**83.** We now pass on to the following theory of the synchronous motor, due to the author.[1]

We consider the case of an alternating-current machine whose field is separately excited by a direct current, while a simple alternating current passes round the armature.

Let $w$ = output of motor in watts.
$i$ = R.M.S. value of armature current.
$r$ = resistance of armature.
$E$ = R.M.S. P.D. impressed between motor terminals.
$e$ = R.M.S. Counter E.M.F. of motor.
$L$ = coefficient of self-induction of armature.
$n$ = frequency of armature current.

[1] *Philosophical Magazine and Electrical Review*, vol. 36, Nos. 905, 913; vol. 37, Nos. 919, 924, 926, 929, 934, 938; and vol. 38, Nos. 949, 974.

$I$ = impedance of armature $= \sqrt{\{r^2 + (2\pi nL)^2\}}$.
$S$ = reactance of armature $= 2\pi nL$.
$\psi$ = phase difference between $i$ and $E$.
$\phi$ = " " $i$ " $e$.
$\theta$ = " " $i$ " $Ii$.

Then the input $= w + i^2r$
and also $= iE \cos \psi$

therefore—

$$w + i^2r = iE \cos \psi \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Solving this equation for $i$, we get—

$$i = \frac{E \cos \psi}{2r} \pm \frac{1}{2r}\sqrt{(E^2 \cos^2 \psi - 4wr)} \quad . \quad . \quad (2)$$

Since $i$ is always real, we must have—

$$E^2 \cos^2 \psi \text{ greater than, or equal to, } 4wr$$

therefore the maximum output is given by—

$$w = \frac{E^2}{4r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This occurs when $\psi = 0$; that is, when the current is in phase with the impressed P.D.

The current corresponding to maximum output is then seen to be—

$$i = \frac{E}{2r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

To obtain the corresponding value of $e$ we proceed as follows:—

There are three E.M.F.s—$e$, $E$, and $Si$—which have a resultant $ri$. Of these $E$ is in phase with $i$, whilst $Si$ is at right angles to it, and $e$ differs in phase with $i$ by an angle $\phi$. The components of $e$ along and at right angles to $i$ are—$e \cos \phi$ and $e \sin \phi$; therefore we must have—

$$E - e \cos \phi = ri$$

and—

$$e \sin \phi = Si$$

But when the output is a maximum—

$$E = 2ri$$

therefore—

$$e \cos \phi = ri$$

whence—

$$\begin{aligned} e &= \sqrt{(r^2 + S^2)}i \\ &= Ii \\ &= \frac{IE}{2r} \quad \ldots \ldots \ldots \quad (5) \end{aligned}$$

Thus at maximum output we have—

$$\left. \begin{aligned} w &= \frac{E^2}{4r} \\ i &= \frac{E}{2r} \\ e &= \frac{IE}{2r} \end{aligned} \right\} \quad \ldots \ldots \ldots \quad (6)$$

It follows that $e$ is greater than, equal to, or less than $E$, according as $I$ is greater than, equal to, or less than $2r$; that is, the P.D. impressed between the motor terminals need not necessarily be greater than the counter E.M.F. developed by the motor.

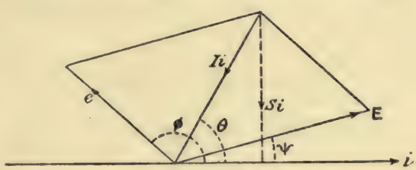


FIG. 42.

A graphic representation of the general phase relationships is given in Fig. 42. In the case of maximum output $\psi$ is zero.

## LIGHT LOAD.

**84.** In the case of the motor running on light load, we may, if we neglect the friction of the bearings, etc., put $w = 0$, so that by (1)—

$$i^2r = iE \cos \psi$$

and—

$$ie \cos \phi = 0$$

This shows that—

$$\phi = \pm \frac{\pi}{2}$$

Also the maximum current at light load is (putting $\psi = 0$) given by—

$$i = \frac{E}{r} \quad \ldots \ldots \ldots \quad (7)$$

which is double the current corresponding to the maximum output.

The corresponding value of the counter E.M.F. may be shown to be—

$$e = \pm \frac{ES}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

### Minimum Current at Given Power.

**85.** Taking the equation—

$$w + i^2 r = iE \cos \psi$$

and equating $\frac{di}{d\psi}$ to zero, we see that, whatever be the output of the motor, the current is a minimum (for that output) when $\psi = 0$, that is, when the current is in phase with the impressed P.D.

### Fundamental Equation of Synchronous Motor.

**86.** With notation as above, p. 127, we have—

$$w = ie \cos \phi$$

and—

$$E^2 = e^2 + I^2 i^2 - 2eIi \cos (\phi - \theta) \quad \text{(From Fig. 42, p. 128.)}$$

Also—

$$\cos \theta = \frac{r}{I}; \ \sin \theta = \frac{S}{I}$$

therefore—

$$E^2 - e^2 - I^2 i^2 - 2rw = \pm 2S\sqrt{i^2 e^2 - w^2} \quad . \quad . \quad (9)$$

This equation is called by Steinmetz the "fundamental equation." For a given output and applied P.D. this equation gives the relation between the current and the counter E.M.F. It is, in fact, the equation to the characteristic curve of the machine.

### Plant consisting of an Alternator and Synchronous Motor.

**87.** The above fundamental equation obviously retains the same form if $E$ represents the E.M.F. of the generator, $r$ the total resistance, and $I$ the total impedance of the two armatures and

line of a plant consisting of a simple alternating-current generator and a synchronous motor, $S$ now representing the total reactance of the system.

Let $L$ be the sum of the self-inductions of the two armatures and line, so that $S = 2\pi nL$.

The E.M.F., $ri$, which drives the current is the resultant of $E$, $e$, and $Si$, so that $E$, $e$, $Si$, with $ri$ reversed, form a system of E.M.F.s in equilibrium.

In Fig. 43 let the positive direction of rotation be counter-clockwise, and let $Oi$ be the direction of the current. The instantaneous value of the current does not concern us at present.

Take $OR'$ equal to $ri$ reversed, and, consequently, opposing the current; let $OS' = Si$, lagging behind the current by a quarter of a period. The resultant, $OT'$ of $OR'$ and $OS'$, must then be equal, and opposite to the resultant of $E$ and $e$. If, therefore, we produce $T'O$ to $T$, and make $OT = OT'$, $OT$ will represent in magnitude and direction the resultant of $E$ and $e$. If, now, we are given the magnitudes of $E$ and $e$, we can find their directions by the parallelogram law. Now, two parallelograms can be constructed, having $OT$ as diagonal, and $E$, $e$ as adjacent sides; but, since $E$ is the E.M.F. of the generator, we take that which gives the component of $E$ along $Oi$ in the same sense as the current.

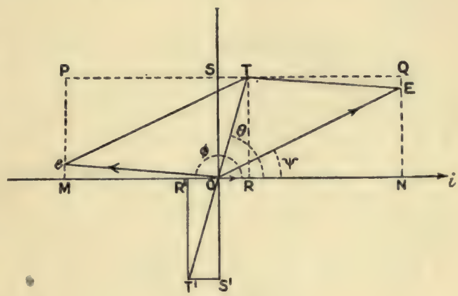

Fig. 43.

The other parallelogram would make $e$ the E.M.F. of the generator. We may notice that the possibility of constructing these two parallelograms proves that, in general, either of two alternating-current machines may be driven as a motor by the other, irrespective of the magnitudes of their relative E.M.F.s.

Let now—

$$\text{angle } iOE = \psi$$
$$\text{angle } iOT = \theta$$

and—

$$\text{angle } iOe = \phi$$

Draw through $T$ the line $PSTQ$ parallel to the line of current, and draw $PM$, $SO$, $TR$, and $QN$ through $e$, $O$, $T$, and $Q$ respectively, at right angles to the line of current. We then have—

$$\tan\theta = \frac{Si}{ri} = \frac{S}{r} = \frac{2\pi nL}{r} \quad . \quad . \quad . \quad . \quad (10)$$

that is, $\theta$ is independent of the current, and $OT$ is a fixed direction relative to $Oi$, so long as the speeds of the machines are kept constant, and $L$ is considered constant.

In Fig. 43, $OS$ (or $PM$) $[= 2\pi nLi]$ is proportional to the current $i$.

$OM$ is the component of $e$ directly opposing $i$. $OR$ is the E.M.F. required to overcome resistance, and $ON$ is the component of $E$ in the direction of $i$; hence rectangle $PSOM$ is proportional to the output of the motor ($w$).

The rectangle $OSTR$ is proportional to the $i^2r$ losses, and the rectangle $OSQN$ is proportional to the output $[iE \cos\psi]$ of the generator.

From this and the equation—

$$w + i^2r = iE\cos\psi$$

it follows that the efficiency of transformation—

$$= \frac{OM}{ON} = \frac{OM}{MR}$$

If the output of the motor is kept constant, we have—

$$\text{rectangle } PSOM = \text{constant}$$

and the locus of $P$ is a rectangular hyperbola having $OM$ and $OS$ as asymptotes (Fig. 44).

Take any point $P$ on this hyperbola. We have seen that $OT$ has a fixed direction relative to $Oi$; and the point $T$ (Fig. 44) on this direction is found by drawing through $P$ a line parallel to $Oi$. Again, $e$ lies on the line through $P$ parallel to $OS$, and $eT = E$ in magnitude. Let the E.M.F. of the generator be kept constant and equal to $E$. With centre $T$, and radius $E$, describe a circle cutting $PM$ in $e$ and $e'$; then the corresponding counter E.M.F. of the motor may be either $Oe$ or $Oe'$, and the current is represented in magnitude by $PM$; that is, corresponding to given values of $E$ and $i$, there are two values of $e$. The relative phases in the two cases are shown in the parallelograms $OeTE$ and $Oe'TE_1'$ (Fig. 44).

To find the point $P'$ on the hyperbola corresponding to minimum current, we have to bring the points $e$ and $e'$ into coincidence. The point $P$ is obviously got by taking $OE'_1$ equal to $E$, and through $E'_1$ drawing $E'_1P'$ parallel to $OT$.

The resulting parallelogram $OP'T_1E_1$ shows that the generator E.M.F. is in phase with the current.

Suppose, now, that the generator field is kept constant, while that of the motor is varied.

When the motor field is small, as *e.g.* $Oe$ (Fig. 44), we see

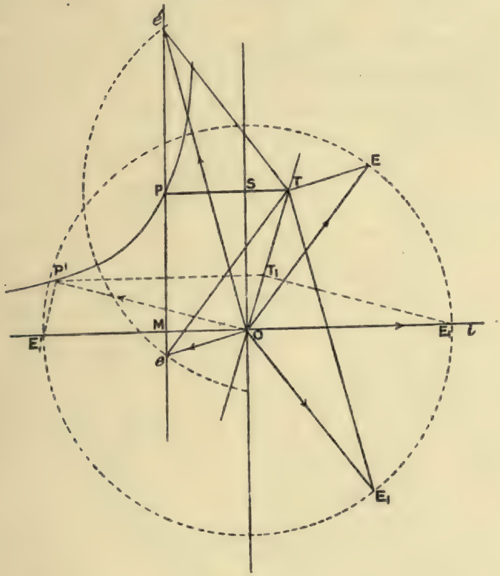


FIG. 44.

that the current leads before the motor E.M.F., and lags behind the generator E.M.F.

When $e$ exactly opposes $i$, the latter lags behind $E$.

When $E$ and $i$ are in phase (minimum current), $i$ lags behind $e$.

When $e$ is still further increased, as *e.g.* $Oe'$, $i$ leads before $E$, and lags behind $e$.

We see, therefore, that by properly adjusting the excitation of the field of the motor, the current may be in phase with, or may lead before, or lag behind, either generator or motor E.M.F.s. This is a point which should be borne in mind, as it has an important bearing on the regulation of a power-transmission plant.

## ARMATURE REACTION.

**88.** We have hitherto supposed that the E.M.F. of the generator, and the counter E.M.F. of the motor, are due simply to the exciting current which passes round their respective field coils, or that the excitations of the fields are fixed by the field currents. Now, this is not quite true, since the magnetization of the fields depends upon all the magnetizing forces in the neighbourhood, and consequently upon the combined effects of the currents in both field and armature coils.

The effect of the armature current in modifying the field is usually called **armature reaction.**

The whole question of armature reaction turns upon the phase relationships of the current and the E.M.F.s of generator and

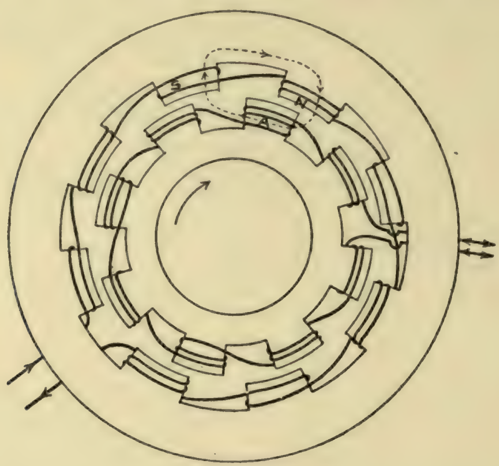

FIG. 45.

motor. Our investigation will include both generator and motor, because both questions are of importance when the whole plant is taken into consideration.

Let Fig. 45 be a diagrammatic representation of an alternating-current machine running as a generator, the direction of rotation being clockwise. If the armature is wound as in figure, the E.M.F. will be a maximum when each armature coil is midway between two consecutive field poles, since the number of lines of force of

the field which thread through the armature coil is then a minimum, and the rate of cutting lines of force is a maximum.

If the current is in phase with the E.M.F., it is evident that the average armature reaction is zero, because the current then attains its maximum value when the coil is midway between two consecutive field-coils, and (the rotation of the armature being clockwise) whatever effect an armature pole, *A* (see Fig. 45), has on a field pole, *S*, as it recedes from it, there will be an equal but opposite effect on the consecutive pole, *N*, on its approach. The same would be true if the difference of phase between E.M.F. and current were 180°, as might be the case if the machine were running as a motor. Thus, in the case of a generator, the average armature reaction is zero if the E.M.F. and armature current are in phase, and in the case of a motor, if the counter E.M.F. and current are in exact opposition.

Now suppose the machine running as a generator and that the current and E.M.F. are not in phase, but that the former leads over the latter by some angle (less than 180°). Consider the same poles in Fig. 45. The current in the coil *A* will now attain its maximum value before it reaches the position midway between *S* and *N*, so that the effect of the current in *A* on the pole *S* will be greater in magnitude than its (opposite) effect on *N*. But the lines of force due to the field current pass through the armature in the direction *N* to *S*, as shown by the dotted lines in the figure, and the lines of force due to the armature current are in the direction from armature to field, from *A* to *S*; thus the effect of the current in *A*, as *A* passes from *S* to *N*, is to strengthen the pole *S*, and weaken the pole *N*, but to a less extent; therefore the average effect of the leading current is to strengthen the field. By similar reasoning we may show that a lagging current weakens the field when the machine is running as a generator.

If the machine were running as a motor, the armature current at any instant would flow in opposition to the counter E.M.F., so that the tendency of the current in *A* would be to weaken the pole *S* and strengthen the pole *N*, and the average effect of a leading armature current is to weaken the field of a motor. Similarly the average effect of a lagging armature current is to strengthen the field of a motor.

We have now proved that the field of a **generator** is **strengthened** when the current **leads** before its E.M.F.,

and **weakened** when the current **lags**; and that the field of a **motor** is **weakened** when the current **leads** before its counter E.M.F., and **strengthened** when the current **lags.**

**89.** Referring to Fig. 44, we conclude that when the excitation of the motor field is small, armature reaction weakens the field of both generator and motor, and when the motor is over-excited both machines have their fields strengthened. When working at minimum current, armature reaction strengthens the motor field, and does not affect the field of the generator. When the field of the motor is unaffected, the generator field is weakened.

**90.** In ordinary working conditions, it is usual to excite the field of the motor to a somewhat greater extent than is required to obtain minimum armature current; for, though the $i^2r$ losses are a minimum and the efficiency a maximum when the current is a minimum, it is advisable to increase the counter E.M.F. to a certain extent in order to cope with accidental variations of the load. Under ordinary working conditions, therefore, the effect of armature reaction is to strengthen the field of the motor and also of the generator, but to a less extent.

**91.** We now proceed to obtain an expression for the alteration, in ampere turns, of the field excitation due to armature reaction.

Let $\phi$ be the displacement of phase of the current over the E.M.F. of the machine; $N$ the number of turns of wire in one section of the armature; $i$ the R.M.S. armature current; $A$ the **mean value of the current** through an angle $\phi$ on each side of its maximum value; and $i_0 \sin pt$ the instantaneous value of the current.

The mean alteration of the field excitation in ampere turns is then—

$$\frac{2AN\phi}{\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where $\phi$ is expressed in circular measure.

But—

$$I = \frac{p}{2\phi} \int_{\left(\frac{\pi}{2} - \phi\right)\frac{1}{p}}^{\left(\frac{\pi}{2} + \phi\right)\frac{1}{p}} i_0 \sin pt \cdot dt$$

$$= \frac{i_0}{\phi} \sin \phi = \frac{\sqrt{2}i}{\phi} \sin \phi \quad . \quad . \quad . \quad . \quad (12)$$

since—

$$i_0 = \sqrt{2}i$$

The required expression for the change of ampere turns is therefore—

$$\frac{2N\phi}{\pi} \cdot \frac{\sqrt{2}i}{\phi} \sin \phi$$

$$= \frac{2\sqrt{2}}{\pi} iN \sin \phi = 0{\cdot}9iN \sin \phi, \text{ nearly} \quad . \quad . \quad (13)$$

To find the total excitation of the field, we must add expression (13) to, or subtract it from, the ampere turns on the field, according as the current leads or lags in a generator, and lags or leads in a motor.

## Stability of a Plant consisting of an Alternating-current Generator and a Synchronous Motor.

**92.** The plant is said to be working in a condition of stability if, for a small increase or diminution of the output of the motor, or for a small increase or diminution of the E.M.F.s of generator or motor, it will continue to work.

We shall suppose the E.M.F. of the generator on open circuit to remain constant, so that the question of stability will involve two distinct problems. (1) Given the E.M.F.s of generator and motor, and the resistance, etc., of the complete circuit consisting of the two armatures and line, when will a breakdown occur if the load on the motor is varied? and (2), given the E.M.F. of the generator, the output of the motor, and the resistance, etc., of the complete circuit, between what limits may the counter E.M.F. of the motor be varied without a breakdown occurring?

The fundamental equation has been obtained in the form—

$$E^2 - e^2 - I^2 i^2 - 2rw = \pm 2S\sqrt{i^2e^2 - w^2}$$

If $E$, $r$, $S$, and $w$ are given, this equation gives a relation between the current $i$ and the counter E.M.F. $e$ of the motor. We can, therefore, plot a curve with values of $e$ for ordinates, and corresponding values of $i$ for abscissæ; and by giving different values to $w$, a series of such curves will be obtained. We may call them **characteristic curves.** A series of characteristic curves is given in Fig. 46.

The fundamental equation is of the fourth degree in $e$ and $i$, and is symmetrical with respect to the axes (since $e$ and $i$ occur with even exponents only).

The curves in the first and third quadrants represent the condition of affairs when the machine is running as a generator, since

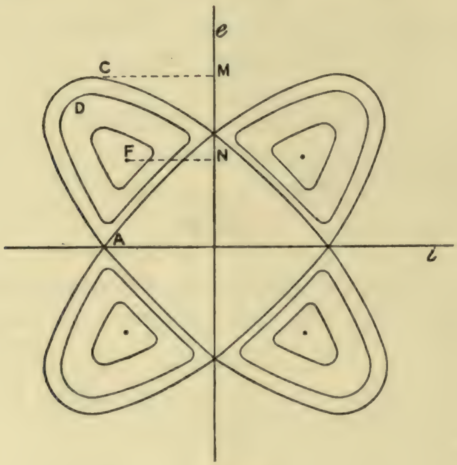


FIG. 46.

in these quadrants $e$ and $i$ have always the same sign. In the second and fourth quadrants the machine is running as a motor. The symmetry of the curves renders a detailed consideration of the second quadrant sufficient.

By squaring and transposing the fundamental equation, we can put it in the form—

$$I^4 i^4 - 2(I^2E^2 - I^2e^2 - 2rI^2w + 2S^2e^2)i^2 + (E^2 - e^2 - 2rw)^2 + 4S^2w^2 = 0$$

Solving this as a quadratic equation in $i^2$, we get—

$$i^2 = \frac{I^2E^2 - I^2e^2 - 2rI^2w + 2S^2e^2}{I^4}$$

$$\pm \frac{1}{I^4}\sqrt{\{(I^2E^2 - I^2e^2 - 2rI^2w + 2S^2e^2)^2 - I^4[(E^2 - e^2 - 2rw)^2 + 4S^2w^2]\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Now, $i$ must be real, therefore $i^2$ must be real and positive.

This gives us as a first condition that the quantity under the square root must be positive, or, in the limit, zero. That is—

$$(I^2E^2 - I^2e^2 - 2rI^2w + 2S^2e^2)^2 \text{ is greater than or equal to } I^4[(E^2 - e^2 - 2rw)^2 + 4S^2w^2]$$

Since—

$$I^2 = r^2 + S^2$$

this reduces to—

$$I^2e^2E^2 \text{ is greater than or equal to } r^2e^4 + 2I^2rwe^2 + I^4w^2$$

or, taking the square root of both sides—

$$IeE \text{ is greater than or equal to } re^2 + I^2w \quad . \quad . \quad (15)$$

The limiting values of $e$, for any given output, $w$, are therefore given by the equation—

$$re^2 - IeE + I^2w = 0 \quad . \quad . \quad . \quad . \quad . \quad (16)$$

provided, at the same time, the condition—

$$I^2E^2 - I^2e^2 - 2rI^2w + 2S^2e^2 \text{ is greater than or equal to zero } (17)$$

is satisfied, since this is necessary in order that $i^2$ may be positive.

It is not difficult to show that (17) is satisfied by all values of $w$ and $e$ which satisfy (15), for, by (15), we have—

$$I^2w \text{ is less than or equal to } IeE - re^2$$

therefore—

$$\begin{aligned} I^2E^2 - I^2e^2 - 2rI^2w + 2S^2e^2 \text{ is greater than } & I^2E^2 - I^2e^2 - 2r(IeE - re^2) \\ & + 2S^2e^2 \\ \text{greater than } & I^2E^2 - 2rIeE + r^2e^2 - S^2e^2 \\ \text{greater than } & (IE - re)^2 + S^2e^2 \end{aligned}$$

Thus, when (15) is satisfied, so also is (17). It will, therefore, be sufficient to see that condition (15) is always satisfied.

Provided, then, that—

$$re^2 - IEe + I^2w \text{ is equal to or less than zero}$$

the current will be real, and we shall have a possible working condition of the plant.

Solving equation (16) we get—

$$e = \frac{IE}{2r} \pm \frac{\sqrt{I^2E^2 - 4I^2rw}}{2r}$$
$$= \frac{I}{2r}\{E \pm \sqrt{E^2 - 4rw}\} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This gives the maximum and minimum values of $e$ when $E$ and $w$ are given.

If we consider $w$ as variable, it also gives the greatest possible value of $e$ for a given value of the generator E.M.F.; this occurs when $w = 0$, and the greatest possible value of $e$ is given by—

$$e = \frac{IE}{r} = E \sec \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where—

$$\tan \theta = \frac{S}{r}$$

We have noticed before (§ 87) that the generator E.M.F. is not necessarily greater than the counter E.M.F. of the motor. We are now in a position to say that, neglecting losses due to friction, hysteresis, etc., a generator of E.M.F., $E$ can drive as a motor any alternating-current machine whose counter E.M.F. lies between the values $O$ and $E \sec \theta$, where $\theta$ is defined by the equation—

$$\tan \theta = \frac{S}{r}$$

$S$ being the total reactance, and $r$ the total resistance of the two armatures and the line.

It will be well to remember that for the above statement to be true, $e$ and $E$ are not the E.M.F.s on an open circuit, but the E.M F.s due to rotation in the resultant fields determined by the exciting currents, together with the armature currents.

Equation (18) also shows that the maximum output of the motor is given by—

$$w = \frac{E^2}{4r}$$

where $E$ is now the total E.M.F. generated by the driving machine, and $r$ is the total resistance of the two armatures and line.

So long, therefore, that the output lies between the values

0 and $\frac{E^2}{4r}$, and the counter E.M.F. lies between the values 0 and $E \sec \theta$, the plant will be in a condition of stable working.

## Efficiency.

**93.** The electrical efficiency of the motor is—

$$\frac{w}{w + i^2r}$$

where $w$ is the output of the motor, $r$ the resistance of its armature, and $i$ the current. To make the efficiency high, the resistance of the armature should be small, and the plant should work in the neighbourhood of minimum current in the line for the given output.

It appears from practical results that 3 horse-power machines can be made to have an electrical efficiency of over 85 per cent., and 6 horse-power machines of over 90 per cent.; also machines have been made to develop their full-rated powers at frequencies of 60 to 80 complete periods per second.

It has been shown that when the motor is excited to a slightly greater extent than that which corresponds to minimum current, the fields of both generator and motor are strengthened by armature reaction, so that it appears advisable for the armatures of both machines to possess a fair amount of self-induction, not only on account of the strengthening of the fields, but also because, in the case of a breakdown, the counter E.M.F. of self-induction would then prevent too large a current flowing, and would diminish the risk of burning up the armatures.

## PROBLEMS ON CHAPTER XIII.

1. A synchronous motor whose armature has a resistance of 0·5 ohm and self-induction 0·025 henry runs with a P.D. of 100 volts between its terminals. What is its maximum output (irrespective of heating limit)? and what is the corresponding current and counter E.M.F., the frequency being 50 periods per second?

*Answer.* Maximum output = 11,250 watts; current = 100 amperes; counter E.M.F. = 1180 volts.

2. A synchronous motor is capable of giving 30 kilowatts without falling out of step when running on a 200-volt 50-frequency circuit, and its counter E.M.F. is then 1500 volts. What is the resistance and self-induction of its armature? and what is the current corresponding to its maximum output?

*Answer.* Armature resistance = 0·333 ohm; armature self-induction = 0·016 henry, nearly; current = 300 amperes.

## CHAPTER XIV.

Polyphase Currents — Generators — Combination of Currents—Production of Rotary Magnetic Fields—Intensity of Rotary Field—Induction Motors—Starting Devices for Induction Motors—Structure and Performance of Induction Motors.

### POLYPHASE CURRENTS.

**94.** In the last chapter we saw that an ordinary alternating-current machine can be driven as a motor, provided that its speed is first brought to synchronism with the alternating current which feeds it. In other words, a synchronous motor such as is described in the preceding chapter is not self-starting, but requires an auxiliary machine to first bring it into synchronism with the current.

This disadvantage caused electrical engineers to seek some new type of alternating-current motor which would start without mechanical aid. Such a motor was at length devised whose working depends upon a suitable combination of alternating currents differing in phase from each other—so called **Polyphase Currents.**

Before describing the motors themselves, we will consider how polyphase currents can be generated, and investigate the electrical and magnetic effects due to their combined action.

The only systems of polyphase currents in practical use are (1) **Di-phase Currents,** or two alternating currents of the same strength and periodicity, but differing in phase by a quarter of a period, and (2) **Tri-phase Currents,** or three alternating currents of the same strength and periodicity, but mutually out of phase with each other by one-third of a period.

### DI-PHASE CURRENTS.

**95.** Suppose that, instead of tapping the armature of a dynamo-electric machine at two points 180° apart only, as in a

simple alternator, it is tapped at **four** equidistant points—that is, each 90° from the next—and that each tapping is carried to a separate collecting-ring on the shaft of the machine, as is shown diagrammatically in Fig. 47.

It is then evident that each pair of connections 180° apart will

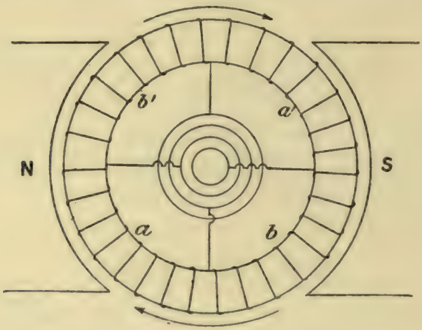


FIG. 47.

give rise to a simple alternating current; but the current in one pair of leads will be 90°, or a quarter of a period, behind the other, according to the direction of rotation of the armature.

In Fig. 47, if the direction of rotation is clockwise, the current in coils $a$, $a'$ will lag behind that in coils $b$, $b'$ by a quarter of a period.

If the current in coils $a$, $a'$ is represented by $i_0 \sin pt$, that in coils $b$, $b'$ will be represented by $i_0 \sin\left(pt + \frac{\pi}{2}\right)$

If these two currents were transmitted along the same wire, the resulting current would be given by—

$$i = i_0 \sin pt + i_0 \sin\left(pt + \frac{\pi}{2}\right)$$

$$= \sqrt{2} i_0 \sin\left(pt + \frac{\pi}{4}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

These are represented graphically in Fig. 48; the dotted line

FIG. 48.

shows the resultant leading before the curve $i_0 \sin pt$ by one-eighth of a period, and lagging behind the curve $i_0 \sin\left(pt + \frac{\pi}{2}\right)$ by the same amount.

## TRI-PHASE CURRENTS.

**96.** If, instead of tapping the armature of an alternator in two places 180° apart, as for monophase currents, or in four places 90° apart, as for di-phase currents, it is tapped at three equidistant points 120° apart, each of the three portions of the armature will be the seat of an alternating E.M.F., but the E.M.F.s will be mutually out of phase by one-third of a period. Each line conductor will therefore carry an alternating current of the same intensity and frequency, but mutually differing in phase by one-third of a period.

Fig. 49 shows diagrammatically such a tri-phase generator in

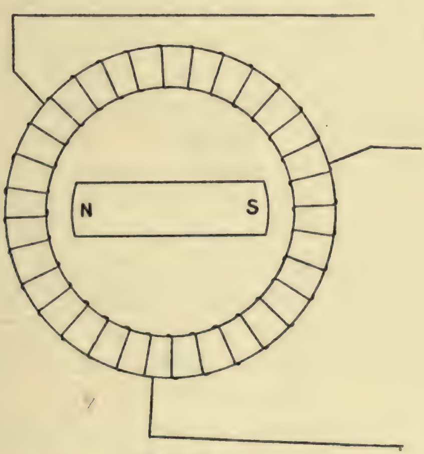


FIG. 49.

which the internal field-magnet rotates whilst the external armature is stationary.

If the current in one of the line wires is represented by $i_0 \sin pt$, those in the other two wires will be $i_0 \sin\left(pt + \frac{2\pi}{3}\right)$ and $i_0 \sin\left(pt + \frac{4\pi}{3}\right)$ respectively.

If these three currents pass simultaneously along the same wire, the resulting current is—

$$i_0 \sin pt + i_0 \sin\left(pt + \frac{2\pi}{3}\right) + i_0 \sin\left(pt + \frac{4\pi}{3}\right) = 0$$

that is, the resultant current is zero when three currents, mutually out of phase by one-third of a period and of the same intensity and frequency, act simultaneously in the same circuit.

## Rotary Magnetic Fields.

**97.** Suppose that an anchor-ring of laminated soft iron is wound as in Fig. 50, and that the coils $a, a'$ are traversed by an alternating current, and the coils $b, b'$ traversed by a second alternating current of the same intensity and frequency, but lagging behind that in $a, a'$ by a quarter of a period. When the current in the coils $a, a'$ is a maximum, that in $b, b'$ is zero, so that if the direction of the current in $a, a'$ at that instant is represented by the arrow heads, the ring will be magnetized in the direction $b, b'$, the south-seeking (S) pole being at $b'$, and the north-seeking (N) at $b$. A quarter of a period later there will be no current in the coils $a, a'$, and that in the coils $b, b'$ will have attained its maximum value, so that the S pole of the ring will now be at $a$, and the N pole at $a'$. In another quarter of a period the S pole will be at $b$, and the N pole at $b'$, and so on.

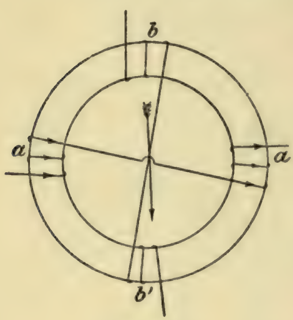


Fig. 50.

We see, therefore, that the anchor-ring, magnetized by two alternating currents differing in phase by a quarter of a period, will have its direction of magnetization rotated at such a speed that it will have turned through 180° in half the periodic time of either current.

A similar rotation of the magnetic axis would occur if the anchor-ring was wound with three circuits fed by three similar alternating currents, differing mutually in phase by one third of a period. In Figs. 51, 52, and 53, $A, B, C$ represent the three windings on an anchor-ring. If the currents in $A, B, C$ are respectively

$i \sin pt$, $i \sin\left(pt + \frac{2\pi}{3}\right)$, and $i \sin\left(pt + \frac{4\pi}{3}\right)$ (see p. 144), the direction of the magnetic field is indicated by the arrows in the cases where the current is zero in $A$, $B$, and $C$ respectively, the arrow head pointing towards the S pole.

The magnetic field, in this case, rotates through a complete revolution in the periodic time of the alternating currents.

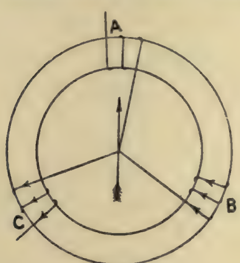


FIG. 51.

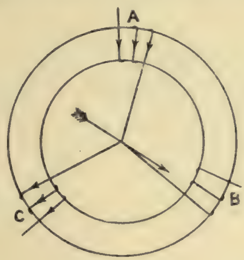


FIG. 52.

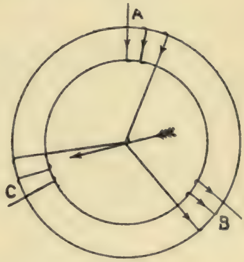


FIG. 53.

If, now, a metal cylinder was placed coaxially within the anchor-ring, and free to rotate round its axis, it is easy to see that rotation would occur, for the rotating magnetic field would induce currents in the body of the cylinder, more or less parallel to its axis, and the mutual action of the rotating field and the field due to the induced currents would give rise to a couple tending to make the cylinder rotate in the same direction as the magnetic field. The couple would be greatest if the induced currents were constrained to flow in directions parallel to the axis of the cylinder. This can be effected by building the cylinder up of thin circular discs of soft iron, and reducing the magnetic reluctance of the circuit by imbedding copper conductors parallel to the axis near the periphery, and connecting them together at the ends. The iron greatly increases the strength of the rotating magnetic field, while the conductors localize the induced currents.

## INDUCTION MOTORS.

**98.** An arrangement such as is here described constitutes an **Induction Motor;** sometimes called a **Rotary Field Motor,** or a **Polyphase Motor.** If the rotating field is produced by **two** alternating currents, we get a **di-phase** motor, and if by three, a **tri-phase** motor.

The stationary part, which is magnetized by the alternating

currents, is called a **Stator;** the rotating part in which currents are induced is called a **Rotor.**

Both Stator and Rotor are built up of laminated iron, and the copper conductors are imbedded in the iron itself so that the air-space between the stationary and moving parts may be made as small as possible.

### Stator Winding.

**99.** There are two general methods of winding the Stator coils; one—**the star-winding**—is depicted in Fig. 54, in the

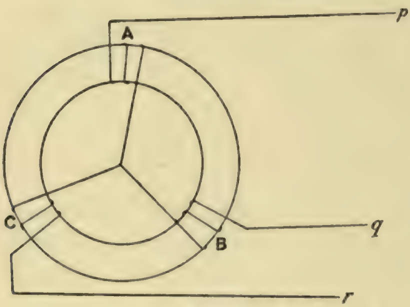


Fig. 54.

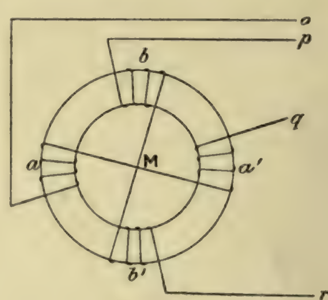


Fig. 55.

case of a tri-phase motor, and in Fig. 55, in the case of a di-phase motor. In this method of winding, all the coils have one end in common, the other end of each going to the respective line wire.

The other way of winding the Stator is to wind the coils like a Gramme ring in one closed coil, tapped at three or four equidistant points, according as it is intended for tri-phase or di-phase currents. This is shown in Figs. 56 and 57, and is termed **Mesh-winding.**

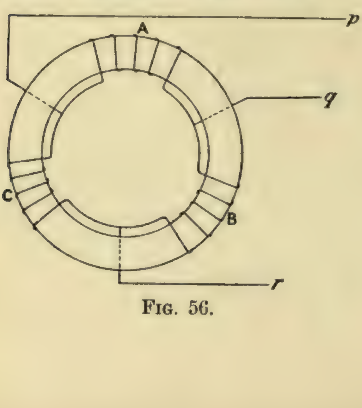


Fig. 56.

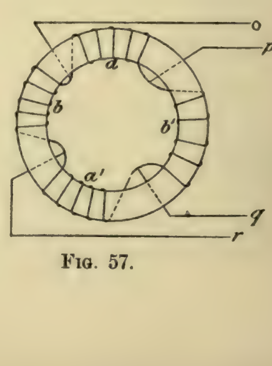


Fig. 57.

## FIELD STRENGTH DUE TO STATOR WINDINGS.

**100.** It will be here assumed that the total induction at any instant is proportional to the corresponding instantaneous value of the ampere turns.

**101. Di-phase Winding.** — Two alternating-currents differing by a quarter of a period, and of equal intensity, are represented in Fig. 58. The current represented by the curve *A* is a

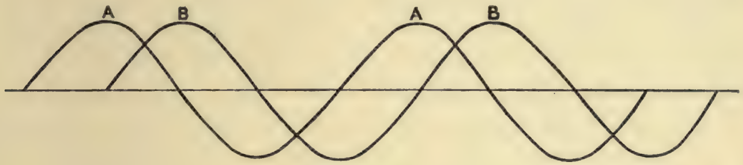


FIG. 58.

quarter of a period in advance of that represented by *B*.

We have now to find the **curve of induction** produced by these currents.

Referring to Figs. 55 and 57, it is seen that the currents in the coils *a*, *a′* **always assist** those in *b*, *b′* in producing the magnetic field; that is, the effects of the currents are algebraically added. Since the number of turns of wire on the stator remains constant, the effect of either current in producing a magnetic field is proportional to the current, so that we may take the current curve itself to represent the induction it produces.

Suppose this to be done for each of the four coils *a*, *a′*, *b*, *b′*. From the way in which these coils are wound in Fig. 55, we see that if the currents in *a*, *b* tend to make the induction clockwise, then those in *a′*, *b′* produce a counter-clockwise induction; the separate inductions are, therefore, represented in Fig. 59 by the curves *B′*, *A*, *B*, *A′*, etc., where the curves *B′*, *A′* are the negative portions of the curves *B*, *A* in Fig. 58 rectified.

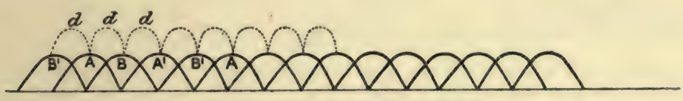


FIG. 59.

The justification for thus rectifying the curves of Fig. 58 lies in the way in which the coils are wound round the Stator. Fig. 59 is not a representation of the currents, but of the induction,

and the phase of the induction depends, not only on the value of the current, but also on the way in which the stator coils are wound.

The **currents** in the coils of Fig. 55 are **two-phased,** but **the induction is four-phased,** for the induction produced by $a'$ differs by 180° from that produced by $a$.

To obtain the resultant, we must add the curves $B'$, $A$, $B$, $A'$ together. We thus obtain the dotted curves $d$ as a graphical representation of the magnitude of the induction at any time. When the current $A$ is zero, $B$ is a maximum. Let the induction then be represented by unity; 45° later its value is 2 sin 45°, or $\sqrt{2}$, and so on.

The induction, therefore, varies between limits proportional to 1 and 1·414; that is, there is a variation of about 20·5 per cent.

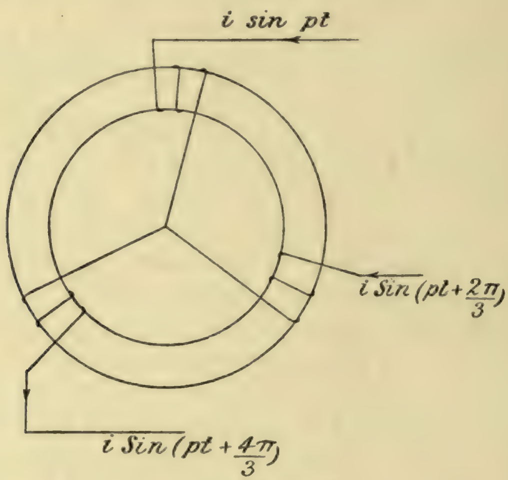


FIG. 60.

from its mean value. This variation, however, becomes considerably toned down by the effect of the rotor currents.

**102. Tri-phase Windings.**—If the stator is wound for tri-phase work, as in Fig. 60, the currents in the three circuits may be taken to be $i \sin pt$, $i \sin\left(pt + \frac{2\pi}{3}\right)$, and $i \sin\left(pt + \frac{4\pi}{3}\right)$ respectively, and are represented graphically in Fig. 61. In order to obtain the curve of induction produced by these three currents flowing through the stator windings, as in Fig. 60, we must rectify

the negative parts of the curves of Fig. 61, and then add all the

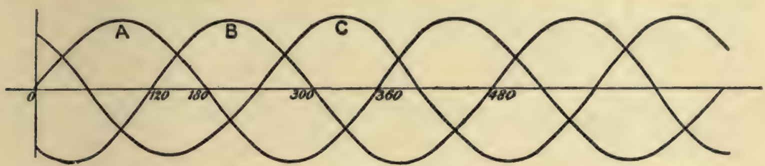


FIG. 61.

curves together. The result of the adding these curves together is to produce the dotted curve in Fig. 62, which represents the value of the induction at any instant.

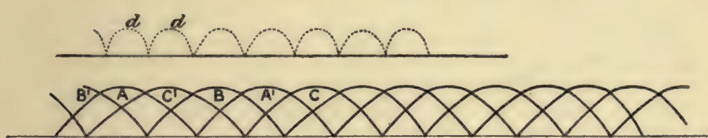


FIG. 62.

From Fig. 62 we see that the maximum value of the induction is proportional to—

$$i + 2i \sin 30^\circ$$

that is, to $2i$; and the minimum value of the induction is proportional to—

$$2i \sin 60^\circ, \text{ or } i\sqrt{3}$$

Thus the induction varies between limits which are proportional to 2 and 1·732, and the variation of the induction from its mean value is about 7 per cent.

Comparing the result with that which was obtained for a diphase induction motor, it is seen that increasing the number of currents round the stator diminishes the percentage variation of the induction from its mean value.

The steady running of an induction motor depends upon the constancy of the torque (or moment of turning couple) of the rotor, and consequently upon the constancy of the strength of the magnetic field in which the rotor revolves. It might be supposed, therefore, that the greater the number of currents, in different phases, exciting the stator, the steadier would be the action of the motor. Practical tests, however, indicate that there is no perceptible difference as regards steady running between a

di-phase and a tri-phase motor. This is probably due to the influence of the rotor currents, which tend to diminish the variations of the induction.

### Difference of Potential between Terminals of Induction Motors.

**103. Di-phase Motor.**—If the coils are joined in the star-grouping as in Fig. 55, with or without a common junction at $M$, the potential difference between the line wires $p$ and $r$, or $o$ and $q$, is the sum of the P.D.s between the terminals of the coils $b$ and $b'$, and equals $2e \sin pt$, if $e \sin pt$ is the E.M.F. absorbed in either coil.

If the coils are star-grouped, and have a common junction $M$, there is a P.D. between $o$ and $p$ given by—

$$v = e \sin pt - e \sin \left(pt - \frac{\pi}{2}\right)$$

$$= \sqrt{2}e \sin \left(pt - \frac{\pi}{4}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Which shows that the P.D. between the two line wires of different phases is $\sqrt{2}$ times the E.M.F. consumed in one of the stator coils, and has a phase midway between the phases of the E.M.F.s in the two corresponding coils.

If the coils are mesh-grouped, as in Fig. 57, the P.D. between $o$ and $p$ is evidently the same as the E.M.F. consumed in $a$, and equals $e \sin pt$.

**104. Tri-phase Motor.**—Let the stator coils be $A$, $B$, $C$, in Figs. 54 and 56, and $p$, $q$, $r$ the line wires. Let the E.M.F.s consumed in $A$ and $B$ be $e \sin pt$ and $e \sin \left(pt - \frac{2\pi}{3}\right)$ respectively.

If the coils are connected in mesh-grouping, as in Fig. 56, the P.D. between $p$ and $q$ is simply the E.M.F. consumed in $A$, or $e \sin pt$.

If, however, the grouping is in the star method, as in Fig. 54, the P.D. between $p$ and $q$ is given by—

$$v = e \sin pt - e \sin \left(pt - \frac{2\pi}{3}\right)$$

$$= \sqrt{3}e \sin \left(pt + \frac{\pi}{4}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

### Currents in the Line Wires.

**105. Di-phase Systems.**—Consider Figs. 55 and 57, and suppose the current strengths in the coils $a$, $a'$, $b$, $b'$, to be the same.

Let the current in $a$, $a'$ be $i \sin pt$, and that in $b$, $b'$, $i \sin\left(pt + \frac{\pi}{2}\right)$.

Then in the star-grouping (Fig. 55) the current in the line wire $o$ is evidently the same as that in the coil $a$.

In the mesh-grouping (Fig. 57) the current in $o$ is the algebraic sum of the currents in $a$ and $b$, and is given by—

$$i' = i \sin pt + i \sin\left(pt + \frac{\pi}{2}\right)$$

$$= \sqrt{2} i \sin\left(pt + \frac{\pi}{4}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

That is, the currents in the line wires are $\sqrt{2}$ times those in the coils, and differ from them in phase by one-eighth of a period.

**106. Tri-phase Systems.**—In a similar manner the currents in the line wires of a tri-phase system are—

Star-grouping: the same as the currents in the stator coils.

Mesh-grouping. The current in line wire, $q$, is given by—

$$i' = i \sin pt - i \sin\left(pt - \frac{2\pi}{3}\right)$$

$$= \sqrt{3} i \sin\left(pt + \frac{\pi}{6}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $i \sin pt$, and $i \sin\left(pt - \frac{2\pi}{3}\right)$ are respectively the currents in coils $A$ and $B$.

### Theory of the Induction Motor.

**107.** The following theory of the induction motor is perfectly general, and independent of the number of phases in the stator and rotor.

Before attacking the problem analytically, it will be well to form a mental picture of what really takes place in an induction motor.

The currents in the stator windings produce a rotating magnetic field. At the instant of switching on the stator currents the rotor

is at rest, so that at the start the currents induced in the rotor coils will have the same frequency as those in the stator. The mutual reaction of the rotor currents and the rotating field produces a torque which, if sufficiently great to overcome the load and the friction of the rotor bearings, will make the rotor follow the magnetic field.

It will then speed up until the torque produced electrically balances that due to the friction and the load. As the speed increases, the rate at which the conductors on the rotor are cut by the rotating magnetic field diminishes, because the rate of cutting is directly proportional to the angular velocity of the field relative to the rotor. In consequence of this, both the magnitude and the frequency of the rotor currents diminish as its speed increases, if the load on the motor is kept constant. If there is no load on the rotor, and if friction is negligibly small, the rotor will keep increasing its speed until it runs synchronously with the rotating field. The rotor currents would then be zero, since the rate of cutting lines of force is zero. As soon as the load is put on the speed is diminished, and again currents are induced in the rotor coils.

From this it is evident that rotary field motors must run at some speed less than that corresponding to synchronism.

The ratio of the frequency of the rotor currents to that of the stator currents is called the **Slip,** and is denoted by the Greek letter $\kappa$.

It will now be seen that an induction motor may be considered as a transformer with a short-circuited secondary circuit free to rotate. The points of difference between the two are, (1) that while the energy given to the secondary of a stationary transformer is expended chiefly in an external circuit, that given to the rotor of an induction motor produces mechanical rotation against an opposing couple; and (2) whereas in a stationary transformer the frequency of the secondary currents is the same as that of the primary currents, the frequency of the rotor currents is, except at the instant of starting, always less than that of the stator currents.

Stationary transformers and induction motors may, in fact, be brought under one theoretical treatment, as has been done by Steinmetz.[1]

[1] "Theory of the General Alternating-current Transformer," by C. P. Steinmetz,

**108.** We will, however, for simplicity, consider the induction motor alone.

Suppose that each circuit in the stator consists of $N_1$ turns of wire, and each rotor circuit of $N_2$ turns; and let the resistance and reactance of each stator circuit be $r_1$ and $s_1$ respectively, and of each rotor circuit, when at rest, $r_2$ and $s_2$ respectively; and let the E.M.F. induced per turn in the stator coils be $e$.

If $\omega_1$ and $\omega_2$ are the angular velocities of the rotating field and rotor respectively, and the frequency of the current $i_1$ in a stator coil be $n$, then the frequency of the current $i_2$ in the rotor coils will be—

$$\frac{\omega_1 - \omega_2}{\omega_1}n$$

or, putting $\kappa$ for the slip—

$$\frac{\omega_1 - \omega_2}{\omega_1}$$

the frequency of the rotor currents is $\kappa n$.

We will suppose that the E.M.F.s and currents have their R.M.S. values.

It follows that the E.M.F. $E_2$ induced per turn in the rotor coils is $\kappa e$. The E.M.F. induced in each circuit of the rotor coils is therefore given by—

$$E_2 = \kappa N_2 e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and the vector equation of E.M.F.s is—

$$r_2 \dot{i}_2 + k\kappa s_2 \dot{i}_2 = \kappa N_2 e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the reactance, when in motion, being $\kappa s_2$. Thus—

$$\dot{i}_2 = \frac{\kappa N_2 e}{r_2 + k\kappa s_2}$$
$$= \frac{(r_2 - k\kappa s_2)\kappa N_2 e}{r_2^2 + \kappa^2 s_2^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The power spent per circuit in heating the rotor is then the scalar product of $i_2$ and $E_2$, that is—

$$\text{Power wasted} = \frac{\kappa^2 N_2^2 e^2 r_2}{r_2^2 + \kappa^2 s_2^2} \quad . \quad . \quad . \quad . \quad (9)$$

*Transactions of the American Institute of Electrical Engineering*, vol. xii. pp. 351–365 (1895); also "Alternating-current Motors," by W. G. Rhodes, *Electrical Review*, vol. xxxvii. pp. 599, 600 (1895); and vol. xxxviii. pp. 139–142 (1896).

Again, the E.M.F. induced in each stator circuit due to the rotating field is given by—

$$E_1 = N_1 e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Also, the current $i_1$ in the stator circuit consists of two parts, the function of one part, $i'_1$, being to excite the stator and produce the rotating field, and that of the other part, $i''_1$, to transmit energy to the rotor.

But—

$$i_1'' = -\frac{N_2}{N_1} i_2$$

which, by (8)—

$$= -\frac{(r_2 - k\kappa s_2)\kappa N_2^2 e}{N_1(r_2^2 + \kappa^2 s_2^2)} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The power transmitted per circuit to the rotor is, therefore, the scalar product of the vectors—

$$-N_1 e$$

and—

$$-\frac{(r_2 - k\kappa s_2)\kappa N_2^2 e}{N_1(r_2^2 + \kappa^2 s_2^2)}$$

that is—

$$\text{Power transmitted to rotor} = \frac{\kappa N_2^2 e^2 r_2}{r_2^2 + \kappa^2 s_2^2} \quad . \quad . \quad . \quad (12)$$

The output, $P$, of the motor is then obtained by subtracting (9) from (12). That is—

$$P = \frac{\kappa N_2^2 e^2 r_2}{r_2^2 + \kappa^2 s_2^2} - \frac{\kappa^2 N_2^2 e^2 r_2}{r_2^2 + \kappa^2 s_2^2}$$

$$= \frac{N_2^2 e^2 r_2 \kappa(1 - \kappa)}{r_2^2 - \kappa^2 s_2^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

To find the torque, $T$, exerted per rotor circuit, we must divide $P$ by the angular velocity $\omega_2$, but—

$$\kappa = \frac{\omega_1 - \omega_2}{\omega_1}$$

therefore—

$$\omega_2 = \omega_1(1 - \kappa)$$

that is—

$$T = \frac{N_2^2 e^2 r_2 \kappa}{\omega_1(r_2^2 + \kappa^2 s_2^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The expression gives the torque in terms of the slip, the resistance and reactance of the rotor coils, the E.M.F. developed per turn by the rotating field at full frequency, the number of turns

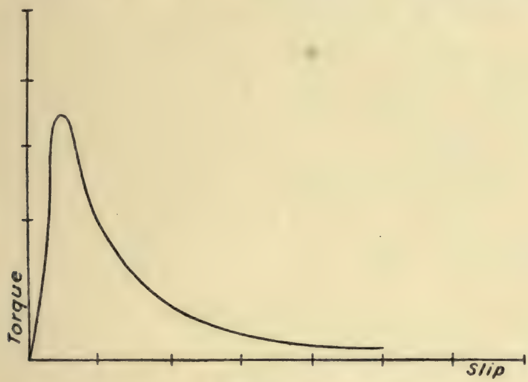


FIG. 63.

per circuit on the rotor, and the angular velocity of the rotating field.

The relation between torque and slip is shown graphically in Fig. 63, which curve has its maximum value when $\kappa = \frac{r}{s}$

Equation (14) shows that the starting torque is (putting $\kappa = 1$)—

$$T_0 = \frac{N_2^2 e^2 r_2}{\omega_1(r_2^2 + s_2^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

which is greater the less the reactance of the rotor-windings, and the less the angular velocity of the rotating field. For variations of $r_2$ the starting torque is the greatest when $r_2 = s_2$, and then becomes—

$$T_0 = \frac{N_2^2 e^2}{2\omega_1 r_2}$$

which varies inversely as the resistance. Thus, to produce a great starting torque, the rotor resistance and reactance should be equal to each other, and each as small as possible.

Also, since $e$ is proportional to the intensity of the rotating fields, it follows that a great torque can only be obtained if, in addition to the other favourable circumstances, the air-gap is made as small as possible, and magnetic leakage as nearly as possible eliminated.

Now, let $Y$ be the admittance per circuit of the stator coils, and $=\sqrt{\rho^2 + \sigma^2}$

where $\rho$ = virtual conductance per circuit of stator,
= ratio of hysteretic energy current to the counter E.M.F. of motor, and
$\sigma$ = susceptance of stator per circuit,
= ratio of magnetizing current to counter E.M.F. of stator.

Also let $r_1$ = resistance of stator per circuit,
$s_1$ = reactance of stator per circuit.

Then the primary exciting current, $i_1'$, is given by—

$$i_1' = -(\rho - k\sigma)N_1 e \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and the total primary current is—

$$\begin{aligned} i_1 &= i_1' + i_1'' \\ &= -(\rho - k\sigma)N_1 e - \frac{\kappa N_2^2 e}{N_1(r_2 + k\kappa s_2)} \\ &= -\left\{\frac{1}{a^2(r_2 + k\kappa s_2)} + \frac{\rho - k\sigma}{\kappa}\right\}\kappa N_1 e \quad . \quad . \quad . \quad (17) \end{aligned}$$

where—

$$a = \frac{N_1}{N_2}$$

The P.D. applied between the stator terminals is—

$$\begin{aligned} E_1' &= -E_1 + i_1(r_1 + ks_1) \\ &= -N_1 e - \left\{\frac{1}{a^2(r_2 + k\kappa s_2)} + \frac{\rho - k\sigma}{\kappa}\right\}(r_1 + ks_1)\kappa N_1 e \\ &= -\left\{1 + \frac{\kappa(r_1 + ks_1)}{a^2(r_2 + k\kappa s_2)} + (\rho - k\sigma)(r_1 + ks_1)\right\} N_1 e \quad (18) \end{aligned}$$

Equations (17) and (18) may be written in the form—

$$i_1 = -\left[\frac{r_2}{a^2(r_2^2 + \kappa^2 s_2^2)} + \frac{\rho}{\kappa} - k\left\{\frac{\kappa s_2}{a^2(r_2^2 + \kappa^2 s_2^2)} + \frac{\sigma}{\kappa}\right\}\right]\kappa N_1 e \quad (19)$$

and—

$$\begin{aligned} E_1' = -\Big[1 + \frac{\kappa(r_1 r_2 + \kappa s_1 s_2)}{a^2(r_2^2 + \kappa^2 s_2^2)} + \rho r_1 + \sigma s_1 + k\Big\{\rho s_1 - \sigma r_1 \\ - \frac{\kappa(\kappa r_1 s_2 - s_1 r_2)}{a^2(r_2^2 + \kappa^2 s_2^2)}\Big\}\Big] N_1 e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20) \end{aligned}$$

Therefore the input of the motor, being the scalar product of $i_1$ and $E_1'$, is given by—

$$W = \left[\left\{\frac{r_2}{a^2(r_2^2 + \kappa^2 s_2^2)} + \frac{\rho}{\kappa}\right\}\left\{1 + \frac{\kappa(r_1 r_2 + \kappa s_1 s_2)}{a^2(r_2^2 + \kappa^2 s_2^2} + \rho r_1 + \sigma s_1\right\}\right.$$
$$\left. - \left\{\frac{\kappa s_2}{a^2(r_2^2 + \kappa^2 s_2^2)} + \frac{\sigma}{\kappa}\right\}\left\{\rho s_1 - \sigma r_1 - \frac{\kappa(\kappa r_1 s_2 - s_1 r_2)}{a^2(r_2^2 + \kappa^2 s_2^2)}\right\}\right]\kappa N_1^2 e^2$$
$$= \left\{\frac{r_2 + \kappa r_1 + 2\rho r_1 r_2 + 2\kappa\sigma s_1 s_2}{a^2(r_2^2 + \kappa^2 s_2^2)} + \frac{\rho}{\kappa} + \frac{r_1}{\kappa}(\rho^2 + \sigma^2)\right\}\kappa N_1^2 e^2 \quad (21)$$

Therefore the efficiency of the motor is given by—

$$\eta = \frac{P}{W}$$
$$= \frac{r_2(1 - \kappa)}{r_2 + \kappa r_1 + 2\rho r_1 r_2 + 2\kappa\sigma s_1 s_2 + \frac{a^2}{\kappa}(\rho + r_1\rho^2 + r_1\sigma^2)(r_2^2 + \kappa^2 s^2)} \quad (22)$$

It is to be noticed from equation (14) that the torque is proportional to the square of $e$. Now $e$ is the E.M.F. induced per turn in the stator winding by the rotating magnetic field; therefore, for a given angular velocity of the rotary field, $e$ is proportional to the field strength. To produce a large torque, it is necessary, therefore, to work at a high induction density in the stator. It follows that a high efficiency and large torque are antagonistic, since with a high induction density the hysteresis loss is large.

## Monophase Induction Motors.

**109.** In the induction motors already described, the torque at any instant is due to a difference between the angular velocity of the magnetic field and that of the rotor.

Suppose that in an induction motor the stator coils form a single circuit fed by a single alternating current, while the rotor is exactly the same as in a rotary field motor. In this the resultant magnetic field preserves a constant **direction,** and simply alternates in **sense** as in a stationary transformer; so that if the rotor is at rest, the alternating currents induced in the rotor coils produce no torque, since the impulses are alternately in opposite directions. If, however, the rotor has, by some means,

an initial velocity given to it, a torque is produced, and the rotor will continue to rotate with a speed which increases till a speed almost corresponding to synchronism with the stator current is attained.

To see this more clearly, we may suppose the alternating magnetic field replaced by two rotary fields of equal strength, and revolving with the same angular velocity in opposite directions. We leave the reader to prove that this is a legitimate supposition.

If the rotor is started by some means in either direction, the induced rotor currents may be regarded as the algebraic sum of the currents due to cutting these two rotary fields.

Let the slips relative to the two rotary fields be $\kappa_1$ and $\kappa_2$ respectively, and suppose that the rotor is started in the direction of $\kappa_2$ diminishing. Then $\kappa_1$ is less than, and $\kappa_2$ greater than, unity, and $\kappa_1 + \kappa_2 = 2$.

Let the corresponding torques be $T_1$ and $T_2$, which act in opposite directions, so that the resultant torque is $T_1 - T_2$; then by (14)—

$$T_1 = \frac{N_2{}^2 e^2 \kappa_1 r_2}{\omega_1(r_2{}^2 + \kappa_1{}^2 s_2{}^2)}$$

and—

$$T_2 = \frac{N_2{}^2 e^2 \kappa_2 r_2}{\omega_1(r_2{}^2 + \kappa_2{}^2 s_2{}^2)}$$

therefore the resultant torque is given by—

$$\begin{aligned} T_1 - T_2 &= \frac{N_2{}^2 e^2 r_2}{\omega_1}\left\{\frac{\kappa_1}{r_2{}^2 + \kappa_1{}^2 s_2{}^2} - \frac{\kappa_2}{r_2{}^2 + \kappa_2{}^2 s_2{}^2}\right\} \\ &= \frac{N_2{}^2 e^2 r_2(\kappa_2 - \kappa_2)(\kappa_1\kappa_2 s_2{}^2 - r_2{}^2)}{\omega_1(r_2{}^2 + \kappa_1{}^2 s_2{}^2)(r_2{}^2 + \kappa_2{}^2 s_2{}^2)} \quad . \quad . \quad . \quad (23) \end{aligned}$$

By supposition, $\kappa_2 - \kappa_1$ is positive; therefore $T_1 - T_2$ is positive, if—

$$\kappa_1\kappa_2 s_2{}^2 - r_2{}^2 \text{ is positive}$$

Now, $\kappa_1\kappa_2$ is never greater than unity; therefore a monophase induction motor cannot run at all unless the reactance is numerically greater than the resistance. So long, however, as $\kappa_1\kappa_2 s_2{}^2 - r_2{}^2$ remains positive, the resultant torque will remain in the same direction, and the motor will continue to run.

It appears, however, that a monophase induction motor cannot

possibly attain synchronism, since that would make $\kappa_1$ or $\kappa_2$ zero, and—

$$\kappa_1\kappa_2 s_2^2 - r_2^2$$

would then be negative. There is, therefore, a limiting speed below synchronism beyond which a monophase motor cannot go.

**110. Starting Devices.**—The difficulty encountered with monophase motors is that they will not start without some auxiliary starting device. The usual starting device is an additional stator winding, round which flows a current out of phase with the main, or running current. It is not necessary to have an additional source of power with which to supply these additional windings with current; they may be arranged in parallel with the main windings on the stator, so that the motor current divides, part going round the main coils, and part round the additional coils.

If the ratio of the reactance to the resistance of the two stator windings is different, the currents in the two circuits will differ in phase, and, by making the difference of phase sufficiently large, a considerable starting-torque will result.

Some makers produce the phase difference in the stator windings by making the self-induction in one circuit large, and introducing capacity into the other circuit by means of condensers made of iron plates in a solution of carbonate of soda.

The splitting of the current causes the magnetic field to rotate, and the motor starts as a di-phase motor. As soon as the rotor is run up to a speed which may be considerably below synchronism, the additional stator circuit is broken; and as there is now a considerable torque exerted on the rotor, the machine will continue to run, and will be able to give out power with a high degree of efficiency.

On referring to equation (23), we find by differentiation with respect to $s_2$, and remembering that $\kappa_1 + \kappa_2 = 2$, that $T_1 - T_2$ is a maximum when—

$$\frac{s_2}{r_2} = \frac{\kappa_1^{\frac{1}{2}}\kappa_2^{\frac{1}{2}} + 2}{\kappa_1^{\frac{3}{2}}\kappa_2^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and that the maximum torque is given by—

$$T_1 - T_2 = \frac{N_2^2 e^2 \kappa_1\kappa_2(\kappa_2^{\frac{1}{2}} - \kappa_1^{\frac{1}{2}})}{\omega_1 r_2} \quad . \quad . \quad . \quad . \quad (25)$$

That is, for a given slip, the maximum torque is inversely propor-

tional to the resistance per circuit of the rotor winding. Thus, while satisfying the relation (23), both the reactance and resistance of the rotor windings should be small if a large torque is desired.

At starting, a monophase motor is acting as a di-phase motor, and, by equation (15), the reactance and resistance of the rotor winding should be equal, and both small, to produce a large starting torque. If, as is usually the case, the reactance is greater than the resistance, then the maximum starting-torque is obtained by increasing the resistance until it numerically equals the reactance. This can be done by winding the rotor with a tri-phase star winding, terminating in collector rings, and by means of three bushes, inserting resistance temporarily in each circuit. The best resistance to insert is that which makes the total resistance per rotor circuit equal to the reactance per circuit.

The reactance, however, causes the rotor currents to lag behind the E.M.F.s producing them. The lagging components of these currents will tend to demagnetize the stator, and a smaller torque results.

The insertion of the resistance cuts down the rotor currents, and brings them more and more nearly into phase with the E.M.F.s induced in the rotor. Wattless current, in either the stator or rotor, are useless; and when occurring in the rotor, are not only useless, but, by their demagnetizing action on the stator, diminish the torque. The author has devised a means of cutting down the wattless rotor currents without interfering with the energy current, by the insertion of E.M.F.s in the rotor circuits, leading a right-angle before the induced E.M.F.s. There is, however, difficulty in obtaining these counter E.M.F.s in exactly the correct phase.

The one defect of the method of inserting non-inductive resistances in the rotor circuits is that the energy current is cut down simultaneously with the wattless currents. A better result is obtained in polyphase induction motors by connecting the secondary circuits of suitable transformers between the several pairs of brushes of the rotor, the primaries being placed in series with the stator windings. Another method consists in inserting the primaries of a tri-phase transformer in the rotor circuits, while the three secondary circuits of the transformer are short-circuited. In each case the rotor is short-circuited on itself when full speed is attained.

Another method employed for producing a high-starting torque in induction motors is as follows: Instead of using non-inductive

resistances in the rotor circuits, a parallel arrangement of a high non-inductive resistance and a low resistance with high inductance is placed in each rotor circuit. No switch is used, as the starting device is always kept in the rotor circuits.

The impedance of a parallel arrangement consisting of a non-inductive resistance $R_1$, and a coil whose resistance is $R_2$, and self-induction $L$, is easily shown to be—

$$\frac{R_1\sqrt{R_2^2 + p^2L^2}}{\sqrt{(R_1 + R_2)^2 + p^2L^2}}$$

where $p = 2\pi$ times the frequency.

This impedance approaches the value $R_1$ when $p$ is large, and the value $\frac{R_1R_2}{R_1 + R_2}$ when $p$ is small.

Now, at the start the rotor currents have the full frequency of the stator currents, so that the external rotor impedance is approximately $R_1$; but when full speed is attained, their frequency is but a small fraction (about 5 per cent.) of that of the stator currents, so that the external rotor impedance now approaches the value $\frac{R_1R_2}{R_1 + R_2}$, which is so small that its inclusion in the rotor circuits makes no perceptible difference in the efficiency when working. The results produced by this starting device are said to be excellent.

## Starting-Torque of Induction Motors in General.

**111.** From the foregoing it is seen that the only difference between monophase and polyphase induction motors is that in the former the stator is wound with two circuits, one of which is cut out when speed is attained; whereas in the latter all the stator circuits are always in use.

Any means of improving the starting-torque of a monophase motor beyond this phase-splitting arrangement in the stator windings is equally applicable to polyphase motors. Equation (15) shows that to produce a good starting-torque in any induction motor the rotor resistance and reactance should both be low, and as nearly equal as possible. Any device by means of which this is effected is equally applicable to monophase and polyphase induction motors. Thus any improvement in the rotors, or rotor windings, can be applied to all types of induction motors.

## Structure of Induction Motors.

**112.** The stators of all induction motors are built of soft-iron stampings varying from 12 to 20 mils. in thickness. The running coils of monophase induction motors are wound in half-closed slots, which are placed close to the inside periphery, as shown in Fig. 64, while the starting coils are wound in completely closed slots. The stator stampings are built up inside a cast-iron case,

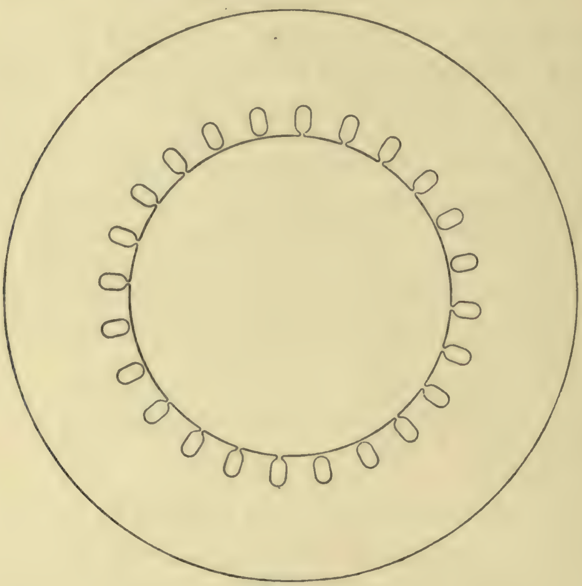

Fig. 64.

to which end shields are bolted. The slots are insulated with micanite or other insulating material. In monophase motors there are two distinct windings, one being the running winding—always in circuit,—the other, the starting winding, in circuit only at starting, and whose function is to provide a cross-magnetization, the action of which on the rotor currents produces the starting-torque.

In di-phase motors there are two similar distinct windings, both of which remain always in circuit.

In tri-phase motors there are three similar distinct windings, which always remain in circuit.

**Rotors.**—The rotors consist of a slotted and laminated core,

the stampings of which need not be so thin as those of the stator, since the frequency of the flux is there much smaller than in the stator. A rotor stamping is depicted in Fig. 65. The rotor winding sometimes consist simply of bars of copper, short-circuited at

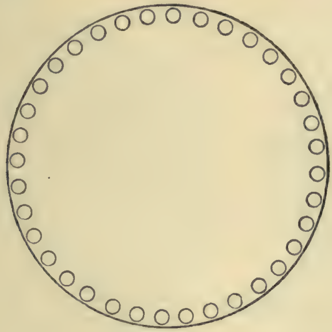

FIG. 65.

each end by heavy copper rings. This is called a **"squirrel-cage"** winding. When a squirrel-cage winding is not used, a tri-phase star winding is almost universally employed, in which case three ends are jointed together, while the remaining three are

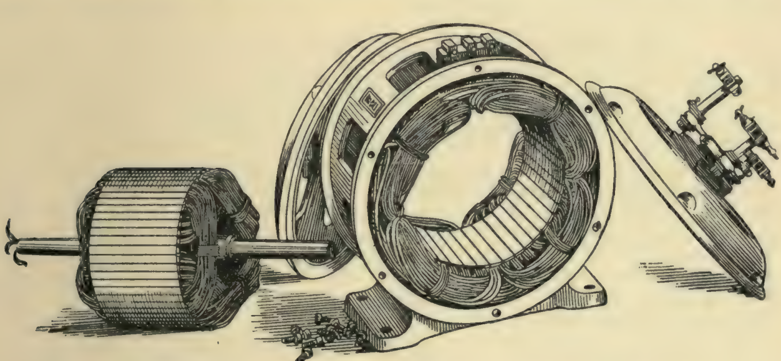

FIG. 66.—The several parts of a Heyland monophase induction motor.

connected to three collector-rings mounted on the shaft. Three brushes rub on these rings, and are held by holders connected to the end shields of the case.

Fig. 66 shows the several parts of a Heyland monophase induction motor, manufactured by Messrs. Witting Bros., London.

Fig. 67 shows a starting resistance and switch, for use with a tri-phase star-wound rotor.

Fig. 68 shows a rotor of a British Thomson-Houston polyphase

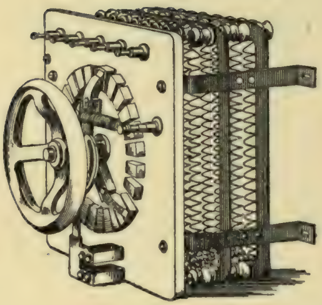

Fig. 67.

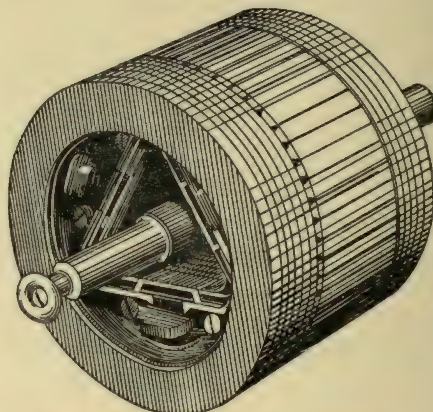

Fig. 68.—Rotor with starting resistance.

induction motor, with a starting resistance inside the spider, and so arranged that it is automatically cut out when speed is got up.

## Performances of Induction Motors.

**113.** The following table gives the results of tests made with monophase motors of the Heyland type:—

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Brake horse-power | ½ | 1 | 3 | 5 | 10 | 20 | 30 | 40 | 50 |
| Full-load speed at 40 cycles per second | 1050 | 1080 | 1080 | 1100 | 750 | 760 | 760 | 575 | 575 |
| Full-load speed at 50 cycles per second | 1310 | 1350 | 1350 | 1380 | 940 | 950 | 950 | 710 | 710 |
| Full-load speed at 60 cycles per second | 1580 | 1620 | 1620 | 1660 | 1130 | 1140 | 1140 | 860 | 860 |
| Number of poles | 4 | 4 | 4 | 4 | 6 | 6 | 6 | 8 | 8 |
| Full-load efficiency per cent. | 65 | 70 | 75 | 80 | 82 | 85 | 87 | 89 | 90 |
| Full-load power factor | 0·65 | 0·70 | 0·73 | 0·75 | 0·80 | 0·82 | 0·83 | 0·85 | 0·85 |
| Full-load current in amperes at 100 volts | 9 | 15 | 41 | 62 | 113 | 215 | 310 | 394 | 580 |
| Approximate weight in lbs. | 145 | 154 | 308 | 440 | 660 | 1100 | 2000 | 2640 | 3500 |

Fig. 69 gives the curves showing the results of a test of a 5 B.H.P. Heyland monophase motor; Fig. 70 gives similar curves for a tri-phase motor manufactured by Messrs. Witting Bros.; while Fig. 71 gives curves relating to a 1·5 B.H.P. monophase motor of the Fuller-Wenström Electrical Manufacturing Company.

From these details we see that induction motors compare favourably with the best shunt-wound direct-current motors,

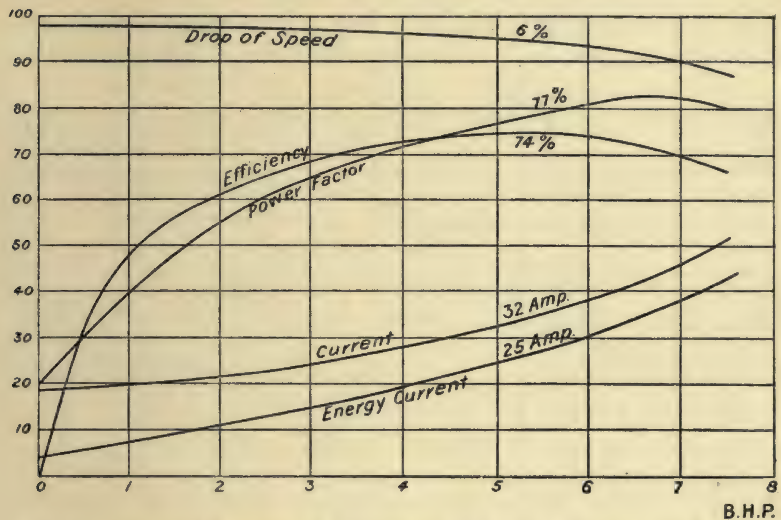


FIG. 69.—Characteristic curve of a brake test for a 5 B.H.P. single-phase motor. 100 cycles. 200 volts. 2000 revs. per min.

both as regards efficiency and constancy of speed under variable loads.

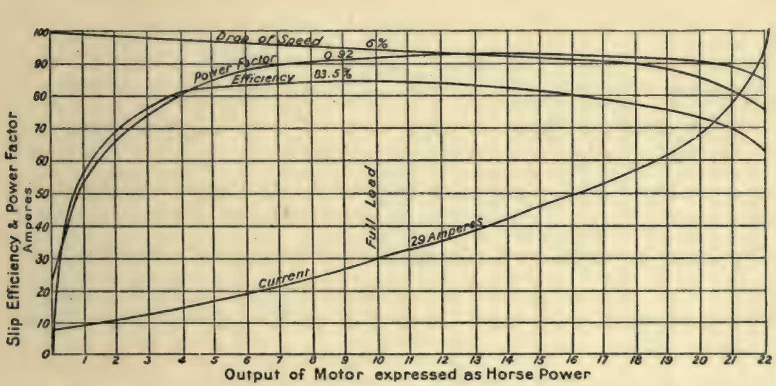


FIG. 70.—Brake test of a 10 B.H.P. three-phase motor. 190 volts. 50 cycles. 1000 revs. per min.

The chief objection to such motors is that no easy means has been devised for varying their speed. It is possible, as has already

been done, to halve the speed by doubling the number of stator-poles, or, generally, to reduce the speed by increasing the number of stator-poles; but such variation is not continuous, and involves complicated windings. The only way of producing a continuous variation of speed is by supplying the motor with current at a

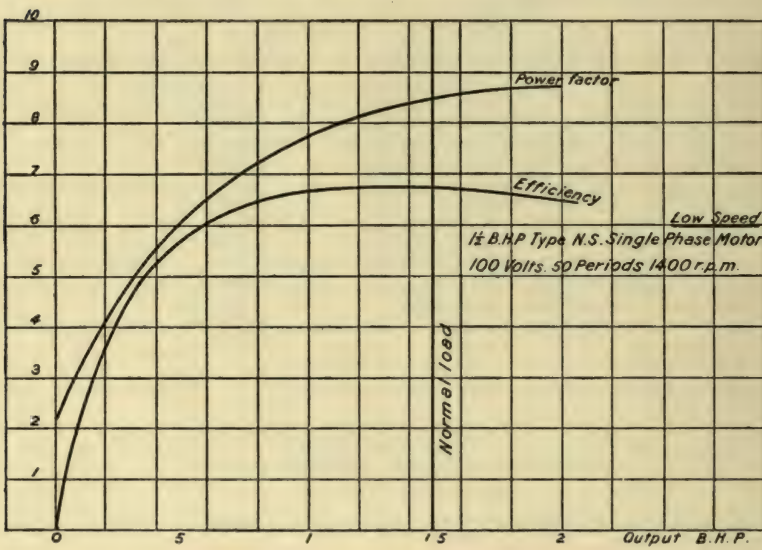


FIG. 71.

varying frequency, for which purpose a **frequency transformer** is required. As, however, no frequency transformer of wide range has hitherto been constructed, a satisfactory solution of the variation of speed of induction motors has not been arrived at. The method of changing the speed by altering the P.D. applied between the terminals of the stator windings is not satisfactory. The P.D. is varied by means of variable resistances, placed in series with the stator windings, entailing a serious waste of energy. An additional objection is that the motor cannot give its full rated output for any length of time when the P.D. is lowered, on account of the heavy currents it requires to do so.

**114.** The noticeable feature of induction motors is their simplicity of structure combined with great mechanical strength. The absence of commutators is an immense advantage, since there is no sparking limit to the output as in direct-current motors;

the consequence is that the output per unit weight is much greater in an induction motor than in a direct current machine. It is not difficult to construct induction motors of a moderate size, which only weigh 60 or 70 lbs. per horse-power, and this without any sacrifice of efficiency.

Unless considerably overloaded, an induction motor will run quite cool compared with direct-current motors of the same weight and output. This is due to the large section which can be given to the conductors and to the fact that the winding is so arranged that the copper losses are not localized, whilst the laminated character of the iron facilitates ventilation.

Although polyphase induction motors are termed **non-synchronous,** it should be borne in mind that there is always a tendency towards synchronism. The speed of an induction motor is, in fact, almost independent of the load; the variation in speed from no-load to full-load seldom exceeds 5 per cent.

It is usual to make large machines multipolar in order to reduce the peripheral velocity of the rotor. Small machines can safely run at 2000 revolutions or more per minute, and so can be made bi-polar. The angular velocity of the rotor, for a given frequency in the stator currents, varies inversely as the number of poles, so that large machines are necessarily of the multipolar type.

**115.** Before concluding this chapter, we draw the attention of the reader to a few points which have to be carefully noted in the design of monophase induction motors.

There are two distinct stator-windings; one—the running coil —must be such as to provide a counter E.M.F. nearly equal to the applied P.D. and capable of carrying the full-load current continuously; the other—the starting-coil—which must likewise provide a counter E.M.F. nearly equal to the applied P.D., but since it is only in circuit for a very short time, it may run at a much higher current density than the running-coil.

Let us examine closely the conditions which the two coils have to satisfy.

When running on load, the power-factor of the motor must be as high as possible. The running-coil, therefore, must be wound so as to have as small an equivalent self-induction as possible; that is to say, the mutual induction between the running coil and the rotor-windings should be as large as possible. The wattless

currents in the stator-windings are solely due to the leakage field, or those magnetic lines which cut the stator coils alone. The running-coil should, therefore, be wound as near to the inside edge of the stator core as possible, and the opening in the slots in which it is wound should be greater than the air-gap between stator and rotor. Further, the winding should be such as to give a fairly high induction density in the iron, otherwise the demagnetizing action of the rotor currents will, at start, have too great an effect, a feeble starting-torque resulting.

The starting-coil, on the contrary, should be wound in closed slots, since in this case a large equivalent self-induction is necessary to give the required difference of phase between the currents in the two stator-windings. Since it is only in circuit for a short time, this coil can have fewer turns than the running-coil, and carry a heavier current.

## CHAPTER XV.

### Polyphase Transformers—Phase Transformers and Rotary Converters.

### Polyphase Transformers.

**116.** We have seen that monophase transformers are used for the purpose of transforming alternating electromotive forces from high to low values, or *vice versâ*.

Polyphase transformers may likewise be used for simply transforming the values of the E.M.F.s, or they may be also arranged so as to transform the number of phases of the E.M.F.s.

There is little to be said in the case of polyphase transformers used simply for transforming E.M.F.s, since the same laws relating to ratio of turns hold good here as in monophase transformers; that is, the ratio between the primary and secondary E.M.F.s is approximately the same as that between the number of primary and secondary turns.

We might, of course, in the case of tri-phase currents, employ three single-phase transformers, viz. one in each of the three circuits; but just as it is unnecessary to have three separate circuits with six line-wires for transmission, so is it equally unnecessary to use three separate transformers. All that is necessary is to have three limbs magnetically short-circuited by common yokes, as shown diagrammatically in Fig. 72, in which $P_1$, $P_2$, $P_3$ represent the primary coils, and $S_1$, $S_2$, $S_3$ the secondaries. The phase relationships between the magnetic fluxes in the three cores will be similar to the phase relationships between the three primary currents.

In the case of di-phase transformers, three cores are again all that is necessary, provided the section of the core wound with the coil which is connected to the common line-wire is $\sqrt{2}$ times the cross-sections of either of the other two, in order that the induction in all three cores may be the same.

Having due regard to these details, the design of a polyphase transformer differs in no respect from that of a monophase transformer.

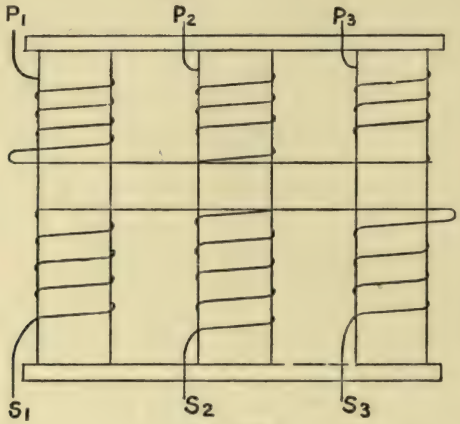


FIG. 72.

## PHASE TRANSFORMERS.

**117.** Phase transformers are used for the purpose of changing alternating current of any number of phases to alternating current of a different number of phases.

The problem of producing such phase transformations was first solved by Professor S. P. Thompson in the following manner:—

A ring transformer having a number of coils in closed series was tapped at three equidistant points, and fed thereat by tri-phase currents, with the result that a rotating magnetic field was produced. On further tapping at the extremities of any diameter, single-phase current could be taken from it; or by tapping at four equidistant points, di-phase currents are obtained. In fact, by making a suitable number of tappings, currents of any number of phases could be obtained.

Such a phase transformer is really an auto-transformer of special type, the correct phase relationships of the secondary currents being dependent upon the production of a magnetic field rotating with uniform angular velocity. It cannot, therefore, be used for transforming monophase currents to polyphase.

In the case in which the primary current is tri-phase, the ratio between the primary and secondary E.M.F.s is approximately

proportional to the ratio of one-third of the whole number of turns to the number of turns between the tappings corresponding to one phase of the secondary.

A method of transforming from di-phase to tri-phase currents was subsequently given by Mr. C. F. Scott, of the Westinghouse Company, by the use of two specially wound transformers, as depicted in Fig. 73.

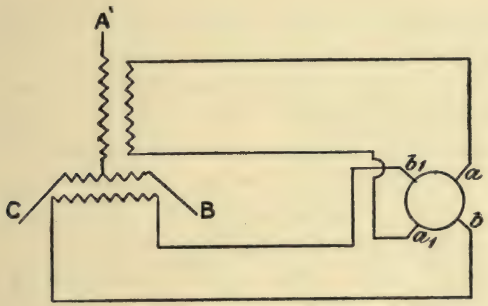


FIG. 73.

The primaries of the transformers are connected to the terminals of a di-phase generator. In the transformers used by Mr. Scott, one secondary *CB* was made equal to 100 turns, and was tapped at its middle point, giving 50 turns on each side. The other secondary had 87 turns ($= 50\sqrt{3}$). One end of it was connected with the middle point of the secondary of the first transformer, and the three free terminals then *A*, *B*, *C*, gave tri-phase E.M.F.s.

The methods given above do not include the most important case from a practical point of view, viz. that of transforming monophase currents to di-phase or tri-phase. Although methods of achieving this have been theoretically propounded, they do not seem to have been realized in practice by any stationary transformer.

## ROTARY CONVERTERS.

**118.** A rotary converter is a machine with but one armature winding, which transforms alternating currents of any number of phases into continuous current, or *vice versâ*.

This definition excludes such combinations as alternating-current motors coupled to direct-current generators, which are called **Motor Generators.**

A rotary converter has less heating and higher efficiency than a motor generator of the same output, one reason for this being that in the converter no mechanical transfer of energy takes place, since the torque required for the generation of the continuous currents and that produced by the alternating current both act on the same armature. In the motor generator, on the other hand, power is transmitted mechanically from the motor to the generator.

A rotary converter is thus a machine with a commutator on one side of the armature, and two, three, or four collector rings on the other side, according as it is intended for mono-phase, tri-phase, or quarter-phase currents. The field magnets are generally excited by a shunt or compound winding on the direct-current side. When the machine is running synchronously with the supply current, the counter-electromotive force in the armature will approximately balance the applied alternating P.D., although they may differ in phase. The E.M.F. developed on the direct-current side will have a steady value equal to the maximum value of the counter E.M.F., and will therefore approximately equal the maximum value of the applied P.D.

**119.** To find the ratio of the E.M.F.s and currents on the two sides of a rotary, we will follow the method employed by Mr. C. P. Steinmetz.[1]

Let Fig. 74 represent diagrammatically the commutator of a direct-current machine, with the armature coils shown connected to adjacent commutator bars. The brushes are represented by $B_1$ and $B_2$, and the field magnets by $F_1$, $F_2$; $a_1$ and $a_2$ are two opposite points of the commutator, and are connected to two collector-rings, $D_1$ and $D_2$.

It is obvious that between the collector-rings there is an alternating E.M.F. $e$, whose maximum value is equal to the continuous E.M.F. $E$, and which makes $p$ periods per revolution of the armature, where $2p$ is the number of poles. In the diagram $p$ is unity.

Hence we have—

$$e = E \sin 2\pi nt$$

where $n$ is the frequency, and $E$ the E.M.F. between the brushes on the commutator.

[1] "The Converter," by C. P. Steinmetz, *Electrical World*, vol. xxxii. pp. 650–652 (1898).

The R.M.S. value of $e$ is, therefore, given by—

$$E_1 = \frac{E}{\sqrt{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

That is, **in a mono-phase rotary converter, the E.M.F. on the direct current side is $\sqrt{2}$ times the R.M.S. value of the applied P.D. on the alternating-current side.**

Again, neglecting losses and difference of phase between the

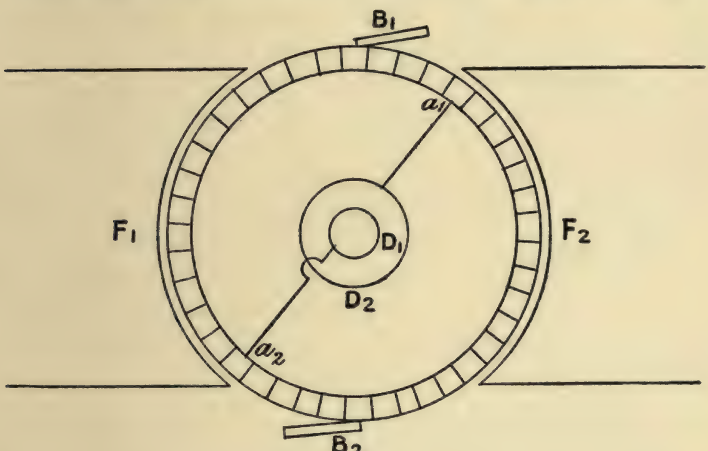


FIG. 74.

applied P.D. and the current, the input is equal to the output, so that if $I$ is the current taken from the commutator, and $I_1$ the R.M.S. value of the supply current, we have—

$$EI = E_1 I_1$$
$$= \frac{E}{\sqrt{2}} I_1 \quad \text{by (1)}$$

therefore—

$$I_1 = \sqrt{2} I \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

That is, **the R.M.S. value of the alternating current is equal to $\sqrt{2}$ times the direct current.**

## $n$-PHASE CONVERTER.

**120.** Suppose now that $n$ equidistant points round the commutator are connected to $n$ collector-rings.

In order to make our calculation clear and intelligible, consider the case in which the armature is star-wound as in Fig. 75, and let $a_1$, $a_2$ be two diametrically opposite coils, connected at $N$, $S$, to two diametrically opposite commutator segments.

It is obvious that if $E$ is the voltage between the commutator brushes, then $E$ is equal to the maximum value of the sum of the alternating E.M.F.s generated in $a_1$ and $a_2$ conjointly; that is, is twice the maximum value of the alternating E.M.F. in $a_1$ alone. It is also obvious that if all the coils are electrically connected at $O$, the result will be the same as if they were not connected there. $O$ is called the neutral point.

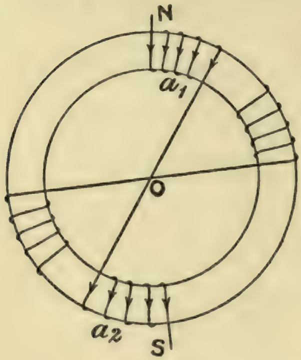

FIG. 75.

We thus see that the maximum value of the E.M.F. generated between the free end of the coil $a_1$ (Fig. 75) and the neutral point is half the E.M.F. between the commutator brushes.

Also the R.M.S. value of the E.M.F. is $\frac{1}{\sqrt{2}}$ times the maximum value; therefore if $E_1$ is the R.M.S. value of the E.M.F. generated between the free end of the coil $a_1$ and the neutral point, we have—

$$E_1 = \frac{E}{2\sqrt{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now revert to the mesh-winding shown in Fig. 76, and let $a_1$, $a_2$, $a_3$ be successive tappings connected to adjacent collector-rings, and let the vectors $OE_1$, $OE_2$, etc. (Fig. 77), represent the R.M.S. values of the E.M.F.s between $a_1$, $a_2$, etc., and the (in this case fictitious) neutral point; then the vector $E_1 E_2$ represents the E.M.F. between two adjacent collector-rings. But the angle $E_1OE_2 = \frac{2\pi}{n}$; therefore—

$$E_1E_2 = 2 \,.\, OE_1 \sin \frac{\pi}{n}$$

$$= \frac{E}{\sqrt{2}} \sin \frac{\pi}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Again, if $I$ is the direct current, and $I_1$ the R.M.S. value of the alternating-current per phase, we have—

$$IE = n \,.\, E_1 I_1$$
$$= \frac{nE}{2\sqrt{2}} \cdot I_1$$

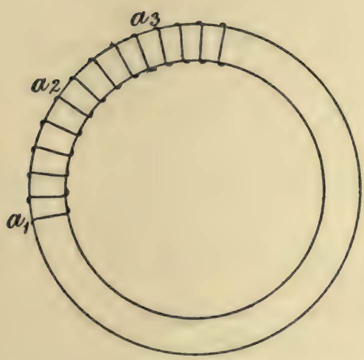


FIG. 76.

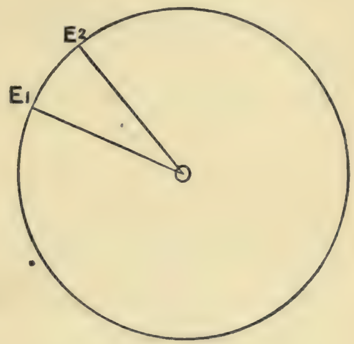


FIG. 77.

therefore—

$$I_1 = \frac{2\sqrt{2} I}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Also the current $I$ between two adjacent lines is given by—

$$IE = \frac{nEI_1'}{\sqrt{2}} \sin \frac{\pi}{n}$$

therefore—

$$I' = \frac{I\sqrt{2}}{n \sin \frac{\pi}{n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

**121.** Applying this to the cases in which the armature is tapped at three and four equidistant points and connected to three and four collector-rings respectively, we get, taking continuous voltage and current as unity—

1st. Tri-phase converter—

Volts between collector-ring and neutral point $= \dfrac{1}{2\sqrt{2}} = 0{\cdot}354$

Volts between adjacent collector-rings $= \dfrac{1}{\sqrt{2}} \sin 60° = \dfrac{\sqrt{3}}{2\sqrt{2}} = 0{\cdot}612$

+ N

$$\text{Amperes per line} = \frac{2\sqrt{2}}{3} = 0{\cdot}943$$

$$\text{Amperes between adjacent lines} = \frac{\sqrt{2}}{3 \sin 60^\circ} = \frac{2\sqrt{2}}{3\sqrt{3}} = 0{\cdot}545$$

2nd. Quarter-phase converter—

$$\text{Volts between collector-ring and neutral point} = \frac{1}{2\sqrt{2}} = 0{\cdot}354$$

$$\text{Volts between adjacent collector-rings} = \frac{1}{\sqrt{2}} \sin 45^\circ = \tfrac{1}{2} = 0{\cdot}5$$

$$\text{Amperes per line} = \frac{2\sqrt{2}}{4} = 0{\cdot}707$$

$$\text{Amperes between adjacent lines} = \frac{\sqrt{2}}{4 \sin 45^\circ} = 0{\cdot}5$$

We can, for reference, tabulate the results thus obtained as follows, R.M.S. values being given throughout:—

| | Continuous current. | Single phase. | Three phase | Quarter phase. | $n$-phase. |
|---|---|---|---|---|---|
| Volts between collector-ring and neutral point . . . . | — | 0·354 | 0·354 | 0·354 | 0·354 |
| Volts between adjacent collector-rings . . . . . . . . | 1 | 0·707 | 0·612 | 0·5 | — |
| Amperes per line . . . . . | 1 | 1·414 | 0·943 | 0·707 | — |
| Amperes between adjacent lines | 1 | 1·414 | 0·545 | 0·5 | — |

The ratios found above between the E.M.F.s on the direct and alternating-current sides of a rotary converter are calculated on the assumption that the induced alternating E.M.F. follows the sine law. This is approximately the case, since the armature of a converter contains a distributed winding.

The ratio between the P.D.s between the commutator-brushes and the collector-rings respectively will, as a rule, not quite agree with the values given above, since there is a drop of volts in the converter armature; and also from the result of armature reaction when the alternating current is not in phase with the P.D. between the collector-rings.

## Heating of Armature of Rotary Converters.

**122.** The current flowing in the armature conductors of a rotary converter is the difference between the currents on the alternating and direct current sides.

Mr. C. P. Steinmetz has shown in the paper referred to above that the ratio of the total energy lost in the armature of an $n$-phase converter with unit power-factor to that lost in the same machine as a direct-current generator at the same output is—

$$\frac{8}{n^2 \sin^2 \frac{\pi}{n}} + 1 - \frac{16}{\pi^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

On giving numerical values to $n$, we see that, taking the heating of the direct-current armature as unity, that of the machines used as a converter is: (1) single phase, 1·37; (2) three phase, 0·555; (3) four phase, 0·37.

When, however, the power factor is not unity, but equals $\cos \phi$, the ratio of the total energy lost in the armature of an $n$-phase converter to that lost in the armature when running as a direct-current generator at the same output is—

$$\frac{8(1 + \tan^2 \phi)}{n^2 \sin^2 \frac{\pi}{n}} + 1 - \frac{16}{\pi^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The proof of these formulæ is given in the Appendix.

We see, therefore, that, excepting the case of the single-phase converter, a greater output can be obtained from a converter than from a generator of the same size for the same permissible rise of temperature.

## PROBLEMS ON CHAPTER XV.

1. What is the amount of energy wasted per second in the armature of a tri-phase rotary converter, the power factor being unity, whose armature resistance is 0·01 ohm, and which gives 100 amperes on the direct-current side?

*Answer.* $56{\cdot}4 \times 10^7$ ergs per second.

2. What is the energy lost per second in the converter in Question 1, if the power factor is $\frac{\sqrt{3}}{2}$, and what if the power factor is $\frac{1}{2}$?

*Answers.* (1) $95{\cdot}9 \times 10^7$ ergs per second; (2) $412 \times 10^7$ ergs per second.

3. The ratio of the watts lost in the armature of an $n$-phase rotary converter with power factor unity to the watts lost in it when running as a direct-current machine, is—

$$\frac{8}{n^2 \sin^2 \frac{\pi}{n}} + 1 - \frac{16}{\pi^2}$$

For what value of $n$ is this a minimum?

*Answer.* $$n = \infty, \text{ or } n = \frac{3\pi}{4} + m\pi$$

where $m$ is any positive integer.

# CHAPTER XVI.

Different Systems of Transmission of Energy—Comparison of Costs of Transmission Lines with Different Systems.

## TRANSMISSION OF ENERGY.

**123.** In the transmission of electrical energy over long distances, it is of importance to consider the cost of the line, which, unless due precaution is taken, may be such as to make the outlay prohibitive.

If energy is transmitted by continuous currents at $E$ volts, we have—

$$EI = W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $W$ is the power in watts transmitted, and $I$ the current in amperes.

It is seen, therefore, that the higher the voltage, the less the current for a given power transmitted. Thus, to economically transmit power over long distances with a reasonable efficiency demands that the voltage of transmission should be as high as possible consistent with other controlling conditions, the chief of which is the quality of the insulation attainable.

**124.** To further illustrate the case, suppose that it is desired to transmit $W$ kilowatts over a distance $x$ centimetres with an efficiency of transmission $\eta$. Let us compare the cost of the transmission lines according as the transmission is effected (1) by continuous currents, (2) by mono-phase alternating currents, (3) by di-phase currents with four wires, (4) by di-phase currents with three wires, one being a common return, (5) by tri-phase currents mesh-grouping, and (6) by tri-phase currents star-grouping.

Since the efficiency is $\eta$, if we let $E$ be the voltage at the generating end of the line, that at the receiving end will be $\eta E$, and the watts lost in the line will be—

$$(1 - \eta)EI \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $I$ is the current in amperes, given by—

$$EI = W$$

in the case of continuous currents, or mono-phase currents with power-factor unity.

Let $R$ be the total resistance of the line; then for two line-wires only—

$$R = \frac{2\rho x}{A}$$

where $A$ is the cross-section of the line conductor, and $\rho$ the specific resistance.

Also, the volume $V$ of the conductor, which we will take to be proportional to the cost, is given by—

$$V = 2Ax$$

therefore—

$$R = \frac{4\rho x^2}{2Ax}$$
$$= \frac{4\rho x^2}{V}$$

We therefore have—

$$(1 - \eta)E = RI$$
$$= \frac{4\rho x^2}{V} \cdot \frac{W}{E}$$

that is—

$$V = \frac{4\rho x^2 W}{(1 - \eta)E^2} \quad \ldots \ldots \quad (3)$$

This shows that the volume of copper in the line-wires for continuous currents, or mono-phase currents with power-factor unity for a given power to be transmitted over a given distance at a given efficiency, is inversely proportional to the square of the voltage of transmission. Thus the volume of copper required for continuous currents and for mono-phase currents is the same.

**125.** In the case of di-phase currents with four wires, it is obvious that the current in each wire is one-half that of the current in a mono-phase system with the same voltage for the same power transmitted, so that the total volume of copper is the same.

When, however, di-phase currents with a common return are

used, and a voltage $E$ between the common return and either of the other two wires, then the current in the common return is $\sqrt{2}I$, where $I$ is the current in either of the other two wires; and, working at the same current density throughout, if the resistance of either outside conductor is $R$, that of the common return is $\frac{R}{\sqrt{2}}$. The loss in the lines is therefore—

$$I^2R + I^2R + 2I^2\frac{R}{\sqrt{2}}$$
$$= I^2R(2 + \sqrt{2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Also the total power transmitted is—

$$2IE$$

If the same power is transmitted by a mono-phase system at the same voltage, with the same loss, the mono-phase current $I'$ is given by—

$$I'E = 2IE$$

or—

$$I' = 2I \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If $R'$ is the resistance of each mono-phase line-wire, we must have—

$$2R'I'^2 = R'I^2(2 + \sqrt{2})$$

that is by (5)—

$$8R'I^2 = I^2R(2 + \sqrt{2})$$

or—

$$R = \frac{8R'}{2 + \sqrt{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Thus each outside wire is $\frac{8}{2 + \sqrt{2}}$ times as large as each mono-phase wire, and the common return $\frac{8\sqrt{2}}{2 + \sqrt{2}}$ times as large. Thus, taking the amount of copper in the mono-phase, or in the direct-current, system as unity, that required for the di-phase system with common return under the same conditions is—

$$\frac{2 + \sqrt{2}}{8} + \frac{2 + \sqrt{2}}{8\sqrt{2}} = 0{\cdot}729$$

or, for the same power transmitted and loss in the line, di-phase

currents required 0·729 of the copper necessary in mono-phase current transmission.

**126.** In the case of a tri-phase installation, mesh-wound, the current in each limb of the mesh is $\frac{1}{\sqrt{3}}$ the current, $I$, in the line (see § 104). The power absorbed in one limb is, therefore, $\frac{1}{\sqrt{3}}IE$ where $E$ is the voltage between the lines. The total power transmitted is, therefore, $\sqrt{3}IE$. Hence, if the same power is transmitted, as by a mono-phase system, the tri-phase current per line-wire is $\frac{I}{\sqrt{3}}$ where $I$ is the mono-phase current. Now let $R$ be the resistance of each of the mono-phase wires, and $R_1$ that of each of the tri-phase wires; then for the same loss—

$$2RI^2 = 3R_1\frac{I^2}{3} = RI^2$$

that is—

$$R_1 = 2R$$

or each of the wires on the tri-phase system has one-half the cross-section of each wire on the mono-phase system. That is, the volume of copper required in the tri-phase system, mesh-wound, is 0·75 that required on the mono-phase system for the same voltage between the lines, the same power transmitted, and the same loss in the lines.

Consider now a tri-phase system star-wound.

In this case the voltage between any two line-wires is $\sqrt{3}$ times the voltage in one limb of the star while the current in any line-wire is the same as that in the corresponding limb.

If, therefore, $E$ is the voltage between any two line-wires, the whole power transmitted is $\sqrt{3}EI$. Hence, as in the case of a mesh-wound system, the amount of copper required is 0·75 times that required in a mono-phase system with the same voltage, distance, efficiency, and power transmitted.

**127.** We can tabulate these results as follows:—

COMPARISON OF AMOUNTS OF COPPER REQUIRED FOR LINE-WIRES FOR TRANSMISSION OF SAME AMOUNT OF POWER, AT THE SAME LINE VOLTAGE OVER EQUAL DISTANCES, AT THE SAME EFFICIENCY.

| | |
|---|---|
| Continuous currents | 1 |
| Mono-phase currents | 1 |
| Di-phase currents with four wires | 1 |
| Di-phase currents with three wires | 0·729 |
| Tri-phase currents with mesh-grouping | 0·75 |
| Tri-phase currents with star-grouping | 0·75 |

**128.** The above table is deduced on the basis of transmitting equal amounts of power over equal distances at equal differences of potential between line-wires and at the same percentage line loss.

If the voltages of transmission are so great that the question of insulation becomes of importance, the basis of comparison has to be altered, since now the limiting factor is the maximum voltage between any two line-wires. Under these conditions, a di-phase three-wire system would require 1·457 times the copper of a mono-phase system, since the potential difference between the two **outside wires** is $\sqrt{2}$ times that between either outside wire and the common return.

## PROBLEMS ON CHAPTER XVI.

1. A tri-phase generator is used to transmit 1000 kilowatts over a distance of 50 miles at 10,000 volts. Calculate the weight of copper in the lines if 20 per cent. of the energy is lost in transmission.

*Answer.* 133 tons, nearly.

2. If the generator in Question 1 has an efficiency of 94 per cent. and feeds a transformer of 98 per cent. efficiency at the other end of the line, what is the efficiency of the system, and what is the output of the transformer?

*Answers.* (1) 73·7 per cent.; (2) 784 kilowatts.

3. What would be the weight of copper in Question 1 if it were mono-phase?

*Answer.* 177 tons, nearly.

## CHAPTER XVII.

Measurement of Power—The Wattmeter — Three-voltmeter Method — Three-ammeter Method—A Dynamometer Method—Measurement of Power in Tri-phase Circuits.

### THE MEASUREMENT OF POWER IN ALTERNATING-CURRENT CIRCUITS.

**129.** We have seen (§ 28) that the power given to an alternating-current circuit is $ei \cos \theta$ where $e$ is the R.M.S. potential difference between its terminals, $i$ the R.M.S. current passing through it, and $\theta$ the difference of phase between the current and potential difference.

It is obvious, therefore, that the power cannot be measured by means of an ammeter and a voltmeter, since the product of their readings is only $ei$.

**130. The Wattmeter.**—The instrument most generally used for the measurement of power is called a Wattmeter, and, as explained in Chap. V., consists generally of two coils of wire placed symmetrically with regard to each other, with their respective planes at right angles to each other. One coil, usually the larger one, is wound with thick wire and is fixed in position, while the other coil, wound with thin wire, is suspended inside the thick wire coil, and is attached to a torsion head by means of a silk thread and a fine phosphor-bronze spiral. The thick coil is placed in series with the circuit, the power given to which is to be determined, and the thin coil is placed in parallel with the circuit. The thin coil should have a very high resistance so as to take a negligibly small fraction of the current in the main circuit, and should have as small a self-induction as possible.

When the wattmeter is in circuit the movable coil rotates through an angle depending upon the product of the field strengths due to the currents in the two coils. The thin coil is brought

back to its original position, with its plane normal to that of the thick coil, by means of the torsion head. The angle through which the torsion head has to be rotated is proportional to the couple necessary to balance the couple due to the mutual action of the two currents, hence—

$$c\theta = i_1 i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $i_1$ is the current in the thick coil and $i$ that in the thin coil and $c$ is a constant.

If the thin coil is non-inductive—

$$e = ri \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $r$ is its resistance, so that $\theta$ is proportional to $ei_1$, that is to the power given to the circuit. If, however, the thin coil is inductive, a correction has to be applied, since then the current $i$ is not in phase with the potential difference between its terminals, and is also less than it would be if it were non-inductive.

We proceed to determine this correcting factor.

**131. Determination of the Correction Factor of a Wattmeter.**—Let $L_1$, $r_1$, $i_1$ be respectively the self-induction, resistance, and R.M.S. current of the circuit, the power given to which is to be measured, and let $L$, $r$, $i$ be respectively the self-induction resistance, and R.M.S. current of the thin coil of the wattmeter, also let $p = 2\pi n$, where $n$ is the frequency of the supply current, and $e$ the R.M.S. potential difference between the common terminals of the two circuits.

Then using vector methods—

$$\dot{i}_1 = \frac{e}{r_1 + kpL_1}$$
$$= \frac{(r_1 - kpL_1)e}{r_1^2 + p^2L_1^2}$$

and—

$$\dot{i} = \frac{(r - kpL)e}{r^2 + p^2L^2}$$

Now, the true power given to the circuit is the scalar product of $i_1$ and $e$, that is—

$$\text{True power} = \frac{e^2 r_1}{r_1^2 + p^2L_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also, the power, as measured by the wattmeter, is the scalar product of $i_1$ and $ri$, that is—

Measured power = scalar product of—

$$\frac{(r_1 - kpL_1)e}{r_1^2 + p^2L_1^2} \text{ and } \frac{r(r - kpL)e}{r^2 + p^2L^2}$$

$$= \frac{r(rr_1 + p^2LL_1)e^2}{(r_1^2 + p^2L_1^2)(r^2 + p^2L^2)} \quad . \quad . \quad . \quad . \quad (4)$$

Therefore—

$$\text{Measured power} \times \frac{r_1(r^2 + p^2L^2)}{r(rr_1 + p^2LL_1)} = \frac{e^2r_1}{r_1^2 + p^2L_1^2}$$

$$= \textbf{true power.}$$

The correcting factor is therefore—

$$\frac{r_1(r^2 + p^2L^2)}{r(rr_1 + p^2LL_1)} = \frac{1 + p^2T^2}{1 + p^2T_1T}$$

where $T = \frac{L}{r}$ and $T_1 = \frac{L_1}{r_1}$ are the time constants of the two circuits respectively.

It is seen from this that if the thin coil of a wattmeter has any self-induction the correction factor depends upon the self-induction of both circuits, unless—

$$1 + p^2T^2 = 1 + p^2T_1T$$

That is, the correcting factor will be unity, if—

$$T(T - T_1) = 0$$

that is, if—

$$T = 0$$

or—

$$T = T_1$$

Since, however, $T_1$ is usually unknown, though it can be determined, it is best to wind the thin coil of the wattmeter so that $T$ is as small as possible. This is done by inserting in series with the thin coil a high non-inductive resistance.

## Other Methods of measuring Power.

**132. Three-voltmeter Method.**—Let, in Fig. 78, $AB$ be an inductive circuit in series with a non-inductive

resistance $BC$. Let $v_1$, $v_2$, and $v$ be the corresponding instantaneous value of the P.D. between the terminals of $AB$, $BC$, and $AC$ respectively. Then we have—

$$v = v_1 + v_2$$

and squaring—

$$v_2 = v_1^2 + v_2^2 + 2v_1v_2$$

therefore—

$$2v_1\frac{v_2}{r} = \frac{1}{r}(v^2 - v_1^2 - v_2^2)$$

where $r$ is the resistance of the circuit $BC$.

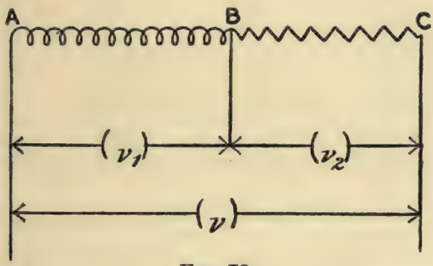


FIG. 78.

Now, $\frac{v^2}{r}$ is the current flowing through $AC$, therefore the power given to $AB$ is—

$$\frac{1}{T}\int_0^T v_1\frac{v_2}{r}dt = \frac{2}{rT}\int_0^T(v^2 - v_1^2 - v_2^2)dt$$
$$= \frac{2}{r}(V^2 - V_1^2 - V_2^2)$$

where $V$, $V_1$, $V_2$ are the R.M.S. value of $v$, $v_1$, and $v_2$ respectively, and $T$ is the periodic time of the currents.

Therefore, the power given to an inductive circuit, $AB$, can be measured by placing a non-inductive resistance, $r$, in series with it, and applying the above formula, which necessitates taking simultaneous readings of three voltmeters. It can be shown that, for the greatest accuracy, it is advisable to make $V_1$ equal to $V_2$.

**133. Three-ammeter Method.**—Let $BC$ (Fig. 79) be the inductive circuit, the power given to which is to be measured. Let $r$ be a non-inductive resistance in parallel with $BC$.

Let $A_1$, $A_2$, $A_3$ be three ammeters arranged as shown, and

giving simultaneous readings $I_1$, $I_2$, $I_3$, and let $i_1$, $i_2$, $i_3$ be corresponding instantaneous values.

Then—

$$i_1 = i_2 + i_3$$

therefore—

$$i_1^2 = i_2^2 + i_3^2 + 2i_2 i_3$$

therefore—

$$r i_2 i_3 = \frac{r}{2}(i_1^2 - i_2^2 - i_3^2)$$

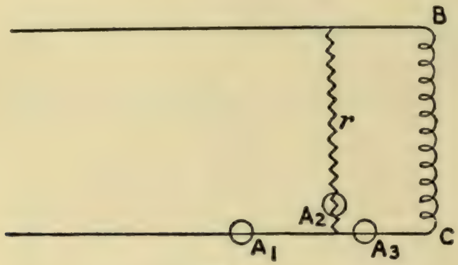

FIG. 79.

But $ri_2$ is the instantaneous P.D. between $B$ and $C$, therefore the power given to $BC$ is—

$$\frac{1}{T}\int_0^T r i_2 i_3 dt$$

where $T$ is the periodic time of the currents. That is, the power given to $BC$ is—

$$\frac{r}{2T}\int_0^T (i_1^2 - i_2^2 - i_3^2)dt$$

$$= \frac{r}{2}(I_1^2 - I_2^2 - I_3^2)$$

Unlike the three-voltmeter method, this method requires extra current, but not additional voltage, and is, therefore, sometimes the more convenient method of the two.

**134. A Dynamometer Method.**—An electrodynamometer is similar in construction to the wattmeter, except that the two coils are constructed of wire of the same thickness, and are usually connected in series, so that the same current flows through both, in which case the dynamometer reading multiplied by a constant gives the mean square value of the current.

The two coils of the dynamometer can, however, be separated, and two independent currents be sent through them. The reading is then proportional to—

$$I_1 I_2 \cos \theta$$

where $I_1$, $I_2$ are the R.M.S. values of the currents, and $\theta$ the phase-difference between them. When used in this way, it is called a split, or divided coil, dynamometer.

Let, in Fig. 80, $BC$ be the inductive circuit, the power given to

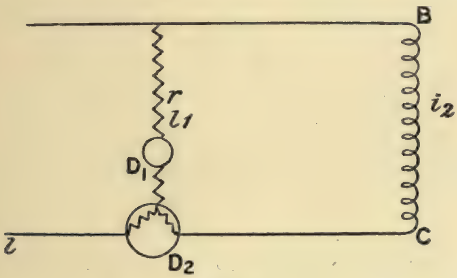


FIG. 80.

which is to be measured. Let $r$ be a non-inductive resistance, $D_1$ a dynamometer, $D_2$ a split dynamometer.

Let $i$, $i_1$, and $i_2$ be the corresponding instantaneous currents in the main circuit, the resistance $r$, and the circuit $BC$ respectively, then—

$$i_1 = i - i_2$$

therefore—

$$ri_1i_2 = rii_2 - ri_2^2$$

But $r$, $i_1$, $i_2$ is the instantaneous power given to $BC$, therefore the power, $W$, given to $BC$ is—

$$W = \frac{1}{T}\int_0^T ri_1i_2dt = \frac{r}{T}\int_0^T i_1i_2dt - \frac{r}{T}\int_0^T i_2^2dt$$
$$= rD_2 - rD_1 = r(D_2 - D_1)$$

where $T$ is the periodic time of the currents, and $D_1$, $D_2$ the readings of the two dynamometers.

## THE MEASUREMENT OF POWER IN TRI-PHASE CIRCUITS.

**135.** The following theory, due to Mr. A. Russell,* gives methods of measuring power in tri-phase circuits :—

**The Measurement of Power in Three-phase Circuits.**—Suppose that there is both a mesh and a star winding as in Fig. 81. Let $a_1$, $a_2$, and $a_3$; $i_1$, $i_2$, and $i_3$; $i_x$, $i_y$, and $i_z$ be the instantaneous values of the currents in the mains, in the mesh winding and in the star winding respectively.

Then—

$$\left.\begin{aligned} a_1 &= i_3 - i_2 + i_x \\ a_2 &= i_1 - i_3 + i_y \\ a_3 &= i_2 - i_1 + i_z \end{aligned}\right\}$$

$$\therefore a_1 + a_2 + a_3 = i_x + i_y + i_z = 0$$

Let $w$ be the instantaneous value of the watts, then—

$$w = v_{1\cdot2}i_3 + v_{2\cdot3}i_1 + v_{3\cdot1}i_2 + v_1 i_x + v_2 i_y + v_3 i_z$$

Now—

$$v_{3\cdot1} = -v_{1\cdot2} - v_{2\cdot3}$$

and—

$$i_y = -i_x - i_z$$

also—

$$v_1 - v_2 = v_{1\cdot3}.$$

$$\therefore w = v_{1\cdot2}(i_3 - i_2 + i_x) + v_{3\cdot2}(i_2 - i_1 + i_z) v_y \quad . \quad (1)$$

By symmetry—

$$\left.\begin{aligned} &= v_{1\cdot2}a_1 + v_{3\cdot2}a_3 \\ &= v_{2\cdot3}a_2 + v_{1\cdot3}a_1 \\ &= v_{3\cdot1}a_3 + v_{2\cdot1}a_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Similarly—

$$w = v_1 a_1 + v_2 a_2 + v_3 a_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The formulæ (2) and (3) give the ordinary methods of measuring power in three-phase circuits.

The first method is to use two wattmeters. The ampere coil of one of them is put in the main $a_1$, and the volt-coil is connected across 1 and 2. The ampere coil of the other is put in the main

* "The Elements of Three-phase Theory," by A. Russell, M A., *Electrician*, vol. xlvii. pp. 639–643, August 16, 1901. The author wishes to express his indebtedness to Mr. A. Russell and The Electrician Printing and Publishing Company, Limited, for kind permission to reproduce the text and diagrams of § 135, and also for the loan of the blocks.

$a_3$, and the volt coil is connected across 3 and 2. Suppose that $w_1$ is the reading on one meter and that $w_2$ is the reading on the

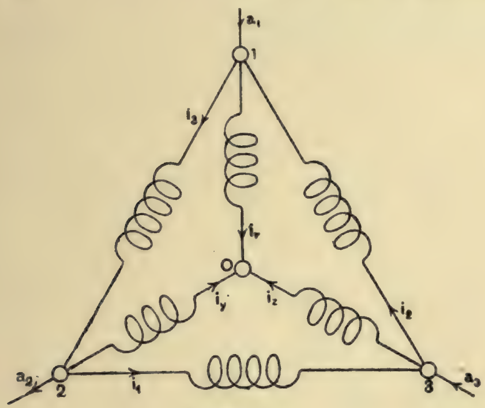


FIG. 81.

other, and suppose that $w_1$ is greater than $w_2$. Then the watts given to the circuit are $w_1 \pm w_2$. If the phase difference between $a_3$ and $v_{3\cdot 2}$ is less than 90°, $w_2$ is positive, but if greater than 90° $w_2$ is negative. It is easy to find out experimentally whether $w_2$ is positive or negative.

The second method is to use three wattmeters, their ampere coils being put in the main circuits and their volt coils across 0 and 1, 0 and 2, and 0 and 3 respectively.

These methods also apply when the windings are as in Fig. 82, where 1, 2, and 3 are the terminals for the mains.

Let $e_1$, $e_2$, and $e_3$ denote the P.D.s between 1 and $L$, 2 and $M$, and 3 and $N$ respectively.

Then—

$$\begin{aligned} w &= e_1a_1 + e_2a_2 + e_3a_3 + v_{\text{LM}}a_1 + v_{\text{NM}}a_3 \\ &= (e_1 - e_2 + v_{\text{LM}})a_1 + (e_3 - e_2 + v_{\text{NM}})a_3 \\ &= v_{1\cdot 2}a_1 + v_{3\cdot 2}a_3 \text{ as before.} \end{aligned}$$

Similarly—

$$\begin{aligned} w &= e_1a_1 + e_2a_2 + e_3a_3 + v_{\text{LO}}a_1 + v_{\text{MO}}a_2 + v_{\text{NO}}a_3 \\ &= v_1a_1 + v_2a_2 + v_3a_3 \end{aligned}$$

It is to be noted that if all the mains are equally loaded, one meter is sufficient. If the volt coil be connected across two of the

mains we multiply the reading by 2. If it be connected from one main to the centre of the system then the multiplying factor is 3.

An important case is when the volt coil of the meter forms one of the arms of a star-box. If the three arms are of equal

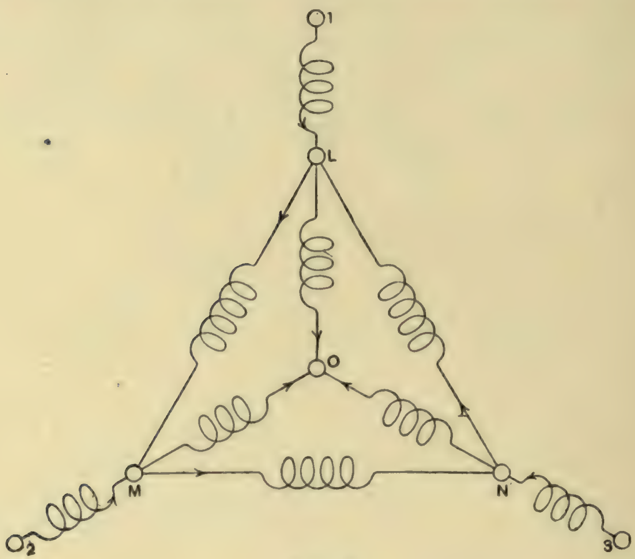


FIG. 82.

resistance, the multiplying factor is 3. If, however, as is often the case in practice, the resistance of the volt coil be different from that of the other two arms, then it is known that the multiplying factor is $2 + \frac{r}{R}$ where $R$ is the resistance of the volt coil and $r$ that of either of the other arms of the box.

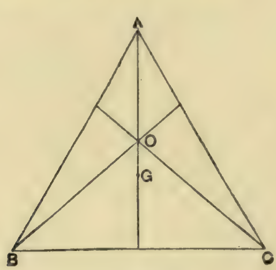


FIG. 83.

This formula can be easily proved as follows: Let $ABC$ (Fig. 83) be the voltage triangle. Then if $O$ be the centre of gravity of masses $\frac{1}{R}$, $\frac{1}{r}$, and $\frac{1}{r}$ placed at $A$, $B$, and $C$ respectively, $OA$, $OB$, and $OC$ will be the three P.D.s to the centre of the star-box.

Also by II. (5)—

$$OA^2\left(\frac{1}{R} + \frac{1}{r} + \frac{1}{r}\right)^2 = \left(\frac{CA^2}{r} + \frac{AB^2}{r}\right)\left(\frac{1}{r} + \frac{1}{r}\right) - \frac{BC^2}{r^2}$$

$$\therefore OA^2\left(2 + \frac{r}{R}\right)^2 = 3 \,.\, AB^2 = 9 \,.\, GA^2$$

where $G$ is the centre of gravity of the triangle $ABC$;

$$\therefore 3\frac{GA}{OA} = 2 + \frac{r}{R}$$

Now, if the arms had been equal $GA$ would be the P.D. across the volt-coil. Hence the required multiplying factor is $3\frac{GA}{OA}$, and this we have shown equals $2 + \frac{r}{R}$.

## MISCELLANEOUS EXAMPLES.

1. If a coil of resistance $r$ and self-induction $L$ is subjected to an alternating P.D. $e$ of varying frequency, and if $n_1$, $n_2$, and $i_1$, $i_2$, are corresponding values of frequency and current, show that

(1) $$r = \frac{e}{i_1 i_2}\sqrt{\frac{n_2^2 i_2^2 - n_1^2 i_1^2}{n_2 - n_1}}$$

(2) $$L = \frac{e}{2\pi i_1 i_2}\sqrt{\frac{i_2^2 - i_1^2}{n_1^2 - n_2^2}}$$

2. If an inductance coil is subjected to an alternating P.D. of varying frequency, and if $T_1$, $T_2$, $T_3$, and $i_1$, $i_2$, $i_3$ are corresponding values of periodic time and current, show that

$$\frac{T_1^2 - T_2^2}{T_2^2 - T_3^2} \cdot \frac{T_3^2}{T_1^2} = \frac{i_1^2 - i_2^2}{i_2^2 - i_3^2} \cdot \frac{i_3^2}{i_1^2}$$

3. An induction motor takes 75 amperes at full load when its power factor is 0·83, what is (1) the energy current and (2) the wattless current?

*Answers.* Energy current = 62·25 amperes.
Wattless current = 41·86 amperes.

4. What is the capacity of a condenser which allows a current of 1 ampere to pass, when a P.D. of 1000 volts, having a frequency of 100 alternations per second, is applied between its terminals?

*Answer.* 1·59 microfarad.

5. If a non-inductive resistance $r$, a resistanceless inductance $L$, and a resistanceless capacity $C$, are placed in parallel on a circuit of frequency $n$, what is (1) the equivalent resistance, (2) the equivalent reactance, (3) the impedance of the arrangement?

*Answers.* (1) Equivalent resistances $= \dfrac{r}{1 + r^2\left(pC - \dfrac{1}{pL}\right)^2}$

(2) Equivalent reactance $= \dfrac{\dfrac{1}{pL} - pC}{\dfrac{1}{r^2} + \left(pC - \dfrac{1}{pL}\right)^2}$

(3) Impedance . . . $= \dfrac{1}{\sqrt{\left\{\dfrac{1}{r^2} + \left(pC - \dfrac{1}{pL}\right)^2\right\}}}$

where $p = 2\pi n$, $n$ being the frequency.

6. Find the current in each branch of the arrangement in Question 15, and also in the main circuit if $pC = \dfrac{1}{pL}$, and if $n = 100$, $r = 10$ ohms, $L = 0{\cdot}05$ henry, and if the applied P.D. is 50 volts.

| | | |
|---|---|---|
| *Answer.* | Current in non-inductive branch | = 5 amperes. |
| | Current in inductive branch | = 1·5915 amperes. |
| | Current in condenser branch | = 1·5915 amperes. |
| | Current in main circuit | = 5 amperes. |

7. An inductive resistance $r_1$, whose self-induction is $L$, is placed in series with a condenser of resistance $r_2$ and capacity $C$ in a circuit of frequency $n$. Find the condition that the P.D. between the condenser terminals may be double of that between the terminals of the inductance.

*Answer.* $$4r_2^2 - r_1^2 = p^2L^2 - \frac{4}{p^2C^2}$$

where $p = 2\pi n$.

8. A non-inductive resistance $r_1$, an inductive resistance $r_2$, whose self-induction is $L$, and a condenser circuit whose resistance is $r_3$, and capacity $C$ are connected (1) in series, and (2) in parallel. What is the power absorbed in the two cases when an alternating P.D. of $e$ volts is applied between the terminals of the arrangements, the frequency being $n$ periods per second?

*Answers.* (1) Power given to series arrangement—

$$= \frac{e^2(r_1 + r_2 + r_3)}{(r_1 + r_2 + r_3)^2 + \left(pL - \dfrac{1}{pC}\right)^2}.$$

(2) Power given to parallel arrangement—

$$= e^2\left\{\frac{1}{r_1} + \frac{r_2}{r_2^2 + p^2L^2} + \frac{r_3}{r_3^2 + \dfrac{1}{p^2C^2}}\right\}$$

where $p = 2\pi n$.

9. A non-inductive resistance of 5 ohms is connected in parallel with a resistanceless inductance of 0·1 henry, and with a capacity of 20 microfarads. If the P.D. is 100 volts, what must be the frequency if the current in the main circuit is (1) 20 amperes and (2) 40 amperes?

*Answers.* (1) 112·5 periods per second.
(2) 4·6 or 2761 periods per second.

10. A non-inductive resistance of 10 ohms is placed in parallel with a resistanceless inductance, the P.D. being 100 volts, and the frequency 50 periods per second. If the main current is 15 amperes, what is the value of the inductance?

*Answer.* 0·02847 henry.

11. A non-inductive resistance of 5 ohms is placed in parallel with a condenser, the P.D. being 100 volts, and the frequency 100 periods per second. If the main current is 21 amperes, what is the value of the capacity?

*Answer.* 102 microfarads.

12. An inductance coil, whose resistance is 5 ohms, and a condenser circuit, whose resistance is 10 ohms, are placed in series on a 50-frequency circuit: the P.D. between the terminals of the condenser circuit is found to be 10 times that between the terminals of the inductance coil. If the inductance is 0·01 henry, what is the value of the capacity?

*Answer.* 54·8 microfarads, nearly.

13. A 4-pole polyphase induction motor running on a 50-frequency circuit, and at a maximum induction in the iron of 14,000 lines per square centimetre, has, when running on full load, a slip of 5 per cent. The dimensions of the stator are: internal diameter 10 inches, external diameter 16 inches, axial length 8 inches, and those of the rotor diameter $9\frac{31}{32}$ inches, axial length 8 inches. Allowing 15 per cent. for the insulation of the stampings, compare (1) the hysteresis losses, (2) the eddy current losses, and (3) the total iron losses in the stator and rotor respectively.

*Answers.* (1) 8 : 1, nearly; (2) 160 : 1, nearly; (3) 116 : 1, nearly.

14. The resistance of a coil of an armature is $7·5 \times 10^{-5}$, and its self-induction is 0·000025 henry, and the current passing through it is 400 amperes. As it passes under the brush a reversing E.M.F. of 2·5 volts is applied to it. How long must the coil be short circuited before a reversal of current takes place?

*Answer.* 0·008 second.

15. If a steady P.D. is applied between the terminals of a circuit whose resistance is 5 ohms, and self-induction 0·005 henry, how long will it take for the current to reach 0·75 of its steady value?

*Answer.* 0·001395 second.

16. If four coils whose self-inductions are 0·1, 0·01, 0·05, and 0·001 henry respectively, and whose resistances are negligible, are connected in parallel, what is the self-induction of the combination?

*Answer.* 0·000885 henry, nearly.

17. If the coils in question 18 have resistances of 10, 1·5, and 0·1 ohms respectively, what is (1) the equivalent resistance, (2) the equivalent self-induction, (3) the impedance of the parallel arrangement, the frequency being 100 periods per second? Take $\pi^2 = 10$.

*Answers.* (1) 0·011256 ohm; (2) 0·000795 henry; (3) 0·57 ohm nearly.

18. What capacity must be placed in (1) parallel, and (2) in series with the arrangement in question 17 to produce resonance?

*Answers.* (1) 3134 microfarads; (2) 3184 microfarads.

19. Two circuits of 100 ohms and 0·1 henry self-induction, and 50 ohms and 20 microfarads respectively, are connected (1) in series and (2) in parallel. Find the total currents, and also the difference of phase between the P.D. and current in the two cases, the P.D. being 1000 volts, and the frequency 50 periods per second.

*Answers.* (1) 5·076 amperes, 40° 20′ lead.
(2) 11·26 amperes, 14° 40′ lead.

20. An induction motor takes 75 amperes on full load, when its power factor is 0·83: what is (1) the energy current, and (2) the wattless current?

*Answers.* Energy current = 62·25 amperes.
Wattless current = 41·85 amperes.

21. What capacity placed in parallel with the induction motor in Question 20 will make the power factor 0·9 on a 500-volt 50 frequency circuit?

*Answer.* 74·44 microfarads, nearly.

22. What capacity placed in parallel with the induction motor in Question 21 will make the power factor unity if the frequency is 100 cycles per second?

*Answer.* 133·2 microfarads, nearly.

# APPENDICES

## APPENDIX A.

**Solution of the equation** (see § 15, Chap. II.)—

$$L\frac{di}{dt} + ri = e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $e$ is constant.

The equation may be written in the form—

$$\frac{di}{dt} + \frac{r}{L}i = \frac{e}{L}$$

Multiplying throughout by $\varepsilon^{\frac{rt}{L}}$, where $\varepsilon$ is the base of the Naperian logarithms, we get—

$$\varepsilon^{\frac{rt}{L}} \cdot \frac{di}{dt} \cdot + \frac{r}{L}i\varepsilon^{\frac{rt}{L}} = \frac{e}{L}\varepsilon^{\frac{rt}{L}}$$

or—

$$\frac{d}{dt}\left\{i\varepsilon^{\frac{rt}{L}}\right\} = \frac{e}{L}\varepsilon^{\frac{rt}{L}}$$

the integral of which is—

$$i\varepsilon^{\frac{rt}{L}} = \frac{e}{r}\varepsilon^{\frac{rt}{L}} + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $C$ is a constant, whence—

$$i = \frac{e}{r} + C\varepsilon^{-\frac{rt}{L}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

To determine the value of $C$, we must remember that $i = 0$ when $t = 0$, therefore from (3)

$$0 = \frac{e}{r} + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Substituting the value of $C$ given by (4) in equation (3) we get—

$$i = \frac{e}{r}(1 - \varepsilon^{-\frac{rt}{L}}) \quad . \; . \; . \; . \; . \; . \; . \quad (5)$$

which gives the value of $i$ at every instant of time.

---

## APPENDIX B.

**Solution of the equation** (see § 21, Chap. III.)—

$$L\frac{di}{dt} + ri = e \sin pt \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

Writing the equation in the form—

$$\varepsilon^{\frac{rt}{L}}\frac{di}{dt} + \frac{r}{L}i\varepsilon^{\frac{rt}{L}} = \frac{e}{L}\varepsilon^{\frac{rt}{L}} \sin pt$$

we get—

$$i\varepsilon^{\frac{rt}{L}} = \frac{e}{L}\int \varepsilon^{\frac{rt}{L}} \sin pt dt \quad . \; . \; . \; . \quad (2)$$

To determine the integral in equation (2) we proceed as follows—
Since—

$$\cos pt + k \sin pt = \varepsilon^{kpt}$$

where $k = \sqrt{-1}$, we have—

$$\begin{aligned}\int \varepsilon^{\frac{rt}{L}}(\cos pt + k \sin pt)dt &= \int \varepsilon^{\frac{rt}{L}} \varepsilon^{kpt} dt \\ &= \int \varepsilon^{\left(\frac{r}{L}+kp\right)t} dt \\ &= \frac{1}{\frac{r}{L} + kp}\varepsilon^{\left(\frac{r}{L}+kp\right)t} + C\end{aligned}$$

where $C$ is a constant—

$$= \frac{\frac{r}{L} - kp}{\frac{r^2}{L^2} + p^2}\varepsilon^{\frac{rt}{L}}(\cos pt + k \sin pt) + C$$

Equating the imaginary parts, we get—

$$\int \varepsilon^{\frac{rt}{L}} \sin pt dt = \frac{\varepsilon^{\frac{rt}{L}}}{\frac{r^2}{L^2} + p^2}\left\{\frac{r}{L} \sin pt - p \cos pt\right\} + C'$$

$C'$ being a constant

$$= \frac{L\varepsilon^{\frac{rt}{L}}}{r^2 + p^2L^2}\{r \sin pt - pL \cos pt\} + C'$$

$$= \frac{L\varepsilon^{\frac{rt}{L}}}{\sqrt{r^2 + p^2L^2}} \sin (pt - \theta) + C' \quad . \quad . \quad . \quad (3)$$

where—

$$\tan \theta = \frac{pL}{r}$$

Substituting this value of the integral in (2) we get, after dividing by $\varepsilon^{\frac{rt}{L}}$—

$$i = \frac{e}{\sqrt{r^2 + p^2L^2}} \sin (pt - \theta) + C\varepsilon^{-\frac{rt}{L}} \quad . \quad . \quad . \quad (4)$$

As $t$ increases $\varepsilon^{-\frac{rt}{L}}$ rapidly diminishes and tends towards the value zero, so that a steady periodic state is given by the equation—

$$i = \frac{e \sin (pt - \theta)}{\sqrt{r^2 + p^2L^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

---

## APPENDIX C.

**Solution of the equation** (see § 24, Chap. IV.)—

$$L\frac{d^2i}{dt^2} + r\frac{di}{dt} + \frac{i}{C} = pe \cos pt \quad . \quad . \quad . \quad . \quad (1)$$

The simplest method of solution here is to assume that $i$ is of the form—

$$i = I \sin (pt - \theta)$$

so that—

$$\frac{d^2i}{dt^2} = -p^2i$$

Equation (1) can then be written—

$$r\frac{di}{dt} + \left(\frac{1}{C} - p^2L\right)i = pe \sin\left(pt + \frac{\pi}{2}\right)$$

and the solution of this, giving the steady periodic state, is by (Appendix B) equation (5)—

$$i = \frac{pe \sin\left(pt + \frac{\pi}{2} - \theta\right)}{\sqrt{\left\{\left(\frac{1}{C} - p^2L\right)^2 + p^2r^2\right\}}}$$

where—

$$\tan\theta = \frac{pr}{\frac{1}{C} - p^2L}$$

or—

$$i = \frac{e \sin(pt - \phi)}{\sqrt{\left\{r^2 + p^2\left(\frac{1}{C} - p^2L\right)^2\right\}}} \quad . \quad . \quad . \quad . \quad (2)$$

where—

$$\tan\phi = \tan\left(\phi - \frac{\pi}{2}\right)$$

$$= \frac{pL - \frac{1}{pC}}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

---

## APPENDIX D.

**Determination of equivalent resistance, reactance, and impedance of *n* mutually reactive circuits in parallel.**

Writing $M_{rr}$ for $s_r$, the equations are (see § 47, Chap. VIII.)—

$$\left.\begin{array}{l}
r_1i_1 + kpM_{11}i_1 + kpM_{12}i_2 + kpM_{13}i_3 + \ . \ . \ . \ + kpM_{1n}i_n = e \\
kpM_{21}i_1 + r_2i_2 + kpM_{22}i_2 + kpM_{23}i_3 + \ . \ . \ . \ + kpM_{2n}i_n = e \\
kpM_{31}i_1 + kpM_{32}i_2 + r_3i_3 + kpM_{33}i_3 + \ . \ . \ . \ + kpM_{3n}i_n = e \\
\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
kpM_{n1}i_1 + kpM_{n2}i_2 + kpM_{n3}i_3 + \ . \ . \ . \ + r_ni_n + kpM_{nn}i_n = e
\end{array}\right\} \quad . \quad (1)$$

Multiplying equations (14) by $k$, and changing signs throughout, we get—

$$\left.\begin{array}{l} (pM_{11} - kr_1)i_1 + pM_{12}i_2 + pM_{13}i_3 + \ldots + pM_{1n}i_n = -ke \\ pM_{21}i_1 + (pM_{22} - kr_2)i_2 + pM_{23}i_3 + \ldots + pM_{2n}i_n = -ke \\ pM_{31}i_1 + pM_{32}i_2 + (pM_{33} - kr_3)i_3 + \ldots + pM_{3n}i_n = -ke \\ \cdots\cdots\cdots\cdots \\ pM_{n1}i_1 + pM_{n2}i_2 + pM_{n3}i_3 + \ldots + (pM_{nn} - kr_n)i_n = -ke \end{array}\right\} . \quad (2)$$

therefore—

$$i_1 = \frac{N}{D}$$

where—

$$N = \left| \begin{array}{l} -ke,\ pM_{12},\ pM_{13}, \ldots\ldots pM_{1n} \\ -ke,\ pM_{22} - kr_2,\ pM_{23}, \ldots pM_{2n} \\ -ke,\ pM_{32},\ pM_{33} - kr_3, \ldots pM_{3n} \\ \cdots\cdots\cdots\cdots \\ -ke,\ pM_{n2},\ pM_{n3}, \ldots pM_{nn} - kr_n \end{array} \right| \quad . \quad . \quad . \quad (3)$$

and—

$$D = \left| \begin{array}{l} pM_{11} - kr_1,\ pM_{12},\ pM_{13}, \ldots pM_{1n} \\ pM_{21},\ pM_{22} - kr_2,\ pM_{23}, \ldots pM_{2n} \\ pM_{31},\ pM_{32},\ pM_{33} - kr_{33}, \ldots pM_{3n} \\ \cdots\cdots\cdots\cdots \\ pM_{n1},\ pM_{n2},\ pM_{n3}, \ldots pM_{nn} - kr_n \end{array} \right| \quad . \quad . \quad . \quad (4)$$

Now—

$$N = -ke \left| \begin{array}{l} 1,\ pM_{12},\ pM_{13}, \ldots\ldots pM_{1n} \\ 1,\ pM_{22} - kr_2,\ pM_{23}, \ldots pM_{2n} \\ 1,\ pM_{32},\ pM_{33} - kr_3, \ldots pM_{3n} \\ \cdots\cdots\cdots\cdots \\ 1,\ pM_{n2},\ pM_{n3}, \ldots pM_{nn} - kr_n \end{array} \right|$$

$$= -ke\{\Delta' - k\Sigma r_s\Delta'_s + k^2\Sigma r_s r_t \Delta'_{st} - \ldots + (-1)^{n-1}k^n r_2 r_3 \ldots r_n\}$$

where—

$$\Delta' = \left| \begin{array}{l} 1,\ pM_{12},\ pM_{13}, \ldots pM_{1n} \\ 1,\ pM_{22},\ pM_{23}, \ldots pM_{2n} \\ 1,\ pM_{32},\ pM_{33}, \ldots pM_{3n} \\ \cdots\cdots\cdots \\ 1,\ pM_{n2},\ pM_{n3}, \ldots pM_{nn} \end{array} \right| \quad . \quad . \quad . \quad (5)$$

and $\Delta'_s$ is $\Delta'$ with the $s$th row and column erased

$\Delta'_{st}$ is $\Delta'$ with both $s$th and $t$th rows and columns erased, and so on

and $\Sigma_{s,r}$ is the sum of all possible products two at a time, and so on.

Put—

$$k^2\Sigma r_s\Delta'_s + k^4\Sigma r_s r_t r_u \Delta'_{stu} + \ldots = A_1$$

and—

$$\Delta' + k^2\Sigma r_s r_t \Delta'_{st} + \ldots = B_1$$

so that—

$$N = e(A_1 - kB_1)$$

Again—

$$D = \Delta - k\Sigma r_s\Delta_s + k^2\Sigma r_s r_t \Delta_{st} - \ldots + (-1)^{n-1}k^n r_1 r_2 \ldots r_n$$

where—

$$\Delta = \begin{vmatrix} pM_{11}, pM_{12}, pM_{13}, \ldots pM_{1n} \\ pM_{21}, pM_{22}, pM_{23}, \ldots pM_{2n} \\ pM_{31}, pM_{32}, pM_{33}, \ldots pM_{3n} \\ \ldots\ldots\ldots \\ pM_{n1}, pM_{n2}, pM_{n3}, \ldots pM_{nn} \end{vmatrix} \quad \ldots \quad (6)$$

$\Delta_s$ is $\Delta$ with the $s$th row and column deleted, and so on.

Put—

$$\Delta + k^2\Sigma r_s r\,\Delta_{st} + \ldots = X$$

and—

$$\Sigma r_s\Delta_s + k^2\Sigma r_s r_t r_u \Delta_{stu} + \ldots = Y$$

therefore—

$$D = X - kY$$

Thus—

$$i_1 = \frac{e(A_1 - kB_1)}{X - kY} \quad \ldots\ldots\ldots \quad (7)$$

Note that $\Delta'$ contains no term in $r_1$.
Let $A_2$, $A_3$, etc.,
$B_2$, $B_3$, etc.,

be the values corresponding to $A_1$, $B_1$ when the 2nd, 3rd, etc., columns only of $\Delta$ are composed of 1's.

Then—

$$i = i_1 + i_2 + i_3 + \ldots + i_n$$
$$= \frac{e\{(A_1 + A_2 + A_3 + \ldots + A_n) - k(B_1 + B_2 + B_3 + \ldots + B_n)\}}{X - kY}$$

Putting—

$$A = A_1 + A_2 + A_3 + \ldots + A_n$$
$$B = B_1 + B_2 + B_3 + \ldots + B_n$$

we get—

$$i = \frac{e(A - kB)}{X - kY}$$

$$= \frac{e(A^2 + B^2)}{XA + YB + k(XB - YA)} \quad . \quad . \quad . \quad . \quad (8)$$

Therefore the equivalent resistance, $R$, and equivalent reactance, $S$, are respectively given by—

$$R = \frac{XA + YB}{A^2 + B^2}$$

$$S = \frac{XB - YA}{A^2 + B^2}$$

and the impedance, $I$, is— . . (9)

$$I = \sqrt{\left\{\frac{(XA + YB)^2 + (XB - YA)}{(A^2 + B^2)^2}\right\}^2}$$

$$= \sqrt{\frac{X^2 + Y^2}{A^2 + B^2}}$$

---

## APPENDIX E.

### Distributed Capacity.

When electrical energy is transmitted over long distances, the capacity of the cables has to be taken into account. The cables act as an infinite number of condensers in parallel.

Let $V$ be the potential at any point $P$ of the cable.

$V + dV$ be the potential at a neighbouring point $Q$.

$i$ be the current at $P$.

$i + d\,i$ be the current at $Q$.

$\rho$ be the resistance per unit length of the cable.

$C$ be the capacity per unit length of the cable $d\,x$ be an element of length and $d\,t$ an element of time.

Then—

$$\frac{dV}{\rho dx} = -i$$

therefore—

$$\frac{dV}{dx} = -\rho i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also—

$$di \,.\, dt = -Cdx \,.\, dV$$

therefore—

$$\frac{di}{dx} = -C\frac{dV}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Differentiating (1) we get—

$$\frac{d^2V}{dx_2} = -\rho\frac{di}{dx}$$

by (2)—

$$= \rho C\frac{dV}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In the case of an infinitely long cable with an applied P.D. $e$ whose periodic time is $T$, the solution is—

$$V = e\epsilon^{-\sqrt{\frac{\rho C\pi}{T}}\,.\,x} \,.\, \sin\frac{2\pi}{T}\left(t - \sqrt{\frac{\rho CT}{4\pi}}\,.\,x\right) \quad . \quad . \quad . \quad . \quad (4)$$

$$i = e\sqrt{\frac{C\pi}{T\rho}}\epsilon^{-\sqrt{\frac{\rho C\pi}{T}}\,.\,x} \,.\, \sin\frac{2\pi}{T}\left(t - \sqrt{\frac{\rho CT}{4\pi}}\,.\,x + \frac{T}{8}\right) \quad (5)$$

where $\epsilon$ is the base of Naperian logarithms and $x$ is the distance of $P$ from the point of application of $e$.

Thus at every point of the cable the current leads one-eighth of a period before the corresponding P.D.

For further information on this point we refer the reader to "Alternating Current Phenomena," by Mr. C. P. Steinmetz, and to "Alternating Currents of Electricity," by Mr. T. H. Blakesley.

---

## APPENDIX F.

### Current in the Armature of an $n$-phase Rotary Converter.

The current in the armature is, for unit power factor (§ 122, Chap. XV.), the difference between the currents on the alternating and continuous current sides respectively.

The alternating current in a section whose axis makes an angle $\theta$ with that of the mean section is given by—

$$i_1 = I_1 \sin(pt - \theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where—

$$I_1 = \frac{2\sqrt{2}}{n \sin \frac{\pi}{n}} I\sqrt{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$I$ being the value of the continuous current between each pair of brushes.

The instantaneous current in the section of the armature is therefore given by—

$$\begin{aligned} i &= i_1 - I \\ &= I\left\{\frac{4 \sin (pt - \theta)}{n \sin \frac{\pi}{n}} - 1\right\} \quad . \quad . \quad . \quad . \quad . \quad (3) \end{aligned}$$

which has a mean square value given by—

$$\begin{aligned} \frac{2}{T}\int_0^{\frac{T}{2}} i^2 dt &= \frac{2}{T}\int_0^{\frac{T}{2}} I^2 \left\{\frac{16 \sin^2 (pt - \theta)}{n^2 \sin^2 \frac{\pi}{n}} - \frac{8 \sin (pt - \theta)}{n \sin \frac{\pi}{n}} + 1\right\} dt \\ &= I^2\left(\frac{8}{n^2 \sin^2 \frac{\pi}{n}} - \frac{16 \cos \theta}{\pi n \sin \frac{\pi}{n}} + 1\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4) \end{aligned}$$

This is a minimum when $\theta = 0$, that is, in the section of the armature situated at the middle of each phase. It is a maximum for $\theta = \pm \frac{\pi}{n}$.

To obtain the mean square current over a complete phase, we have therefore to take the mean value of (4) between the limits $\theta = \frac{\pi}{n}$, and $\theta = -\frac{\pi}{n}$, and obtain the expression—

$$\begin{aligned} &\frac{n}{2\pi}\int_{-\frac{\pi}{n}}^{\frac{\pi}{n}} I^2\left(\frac{8}{n^2 \sin^2 \frac{\pi}{n}} - \frac{16 \cos \theta}{\pi n \sin \frac{\pi}{n}} + 1\right) d\theta \\ &= I^2\left(\frac{8}{n^2 \sin^2 \frac{\pi}{n}} + 1 - \frac{16}{\pi^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (5) \end{aligned}$$

The ratio of the heating of the armature of an $n$-phase rotary

converter to that of the same machine giving the same output as a continuous-current machine is—

$$\frac{8}{n^2 \sin^2 \frac{\pi}{n}} + 1 - \frac{16}{\pi^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For a tri-phase rotary converter this ratio is 0·555, that is, a tri-phase rotary is, neglecting friction and iron losses, capable of giving 80 per cent. greater output for the same heating of the armature than the same machine used as a direct-current generator.

If the power factor is $\cos \phi$, equation (1) becomes—

$$i_1 = I_1 \sin (pt + \phi - \theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the current being a leading one, and the mean square current now is—

$$I^2\left(\frac{8}{n^2 \sin^2 \frac{\pi}{n}} + 1 - \frac{16}{\pi^2} + \frac{8 \tan^2 \phi}{n^2 \sin^2 \frac{\pi}{n}}\right) \quad . \quad . \quad . \quad . \quad (8)$$

In the case of unit power factor, $\phi = 0$, and equation (8) becomes the same as equation (6).

# INDEX

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